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Dwork congruences for reflesive polytopes

Aim: To prove certain congruence properties satisfied by the coefficients of power series which are related to Laurent polynomials whose Newton polyhedra contain 0 as a unique interior lattice point.

Definition: Let p be a prime, $(a(n))_n$ a sequence satisfying $a(n) \in \mathbb{Z}_p$ and $a(0)=1$. We say that $(a(n))$ satisfies the Dwork congruences if for all $m, n, s \geq 0$, one has:

D1: $\dfrac{a(n)}{a(\lfloor \frac{n}{p} \rfloor)} \in \mathbb{Z}_p$

D2: $\dfrac{a(n+mp^{s+1})}{a(\lfloor \frac{n}{p} \rfloor + mp^s)} \equiv \dfrac{a(n)}{a(\lfloor \frac{n}{p} \rfloor)} \mod p^{s+1}$

NB: generally ~~true~~ D3 is implied by D2 for $n < p^{s+1}$

Write $n = n_0 + pn_1 + \dots + p^s n_s$, $0 \leq n \leq p-1$, set $n_{s+1} = m$.

(D2 ⇒) D3: $a(n_0 + n_1 p + \dots + n_{s+1} p^{s+1})\, a(n_1 + \dots + n_s p^{s-1}) \equiv a(n_0 + \dots + n_s p^s)\, a(n_1 + \dots + n_{s+1} p^s)$ $(\mathrm{mod}\ p^{s+1})$

Let $\phi(t) = \sum_{n=0}^{\infty} a(n) t^n$, then the Dwork congruences play a key role in the analytic continuation of $\phi(t)/\phi(t^p)$ to the boundary of the p-adic unit disc.

Thm: (Dwork)

Let $(a(n))$ satisfy D1 and D2, let $\phi^s(t)$ denote the truncated series

$$\phi^s(t) = \sum_{n=0}^{p^s-1} a(n) t^n,$$

let $\mathcal{D} = \{ x \in \mathbb{Z}_p \mid |\phi^1(x)| = 1 \}$. Then $\phi(t)/\phi(t^p)$ is the restriction to $p\mathbb{Z}_p$ of an analytic element ~~fo~~ f of support $\mathcal{D}$,

$$f(x) = \lim_{s\to\infty} \frac{\phi^{s+1}(x)}{\phi^{s}(x^p)} . \qquad (*)$$

Application: Legendre family $X_t : y^2 - x(x-1)(x-t) = 0$, $t \neq 0, 1$.

Let $\phi(t)$ be the hypergeometric series

$$F(\tfrac{1}{2}, \tfrac{1}{2}; 1; t) = \sum_{j=0}^{\infty} \left(\frac{(\frac{1}{2})_j}{j!} \right)^2 t^j$$

Let $t_0 \in \mathbb{F}_p$ be s.t. $\phi^1(t_0) \not\equiv 0 \bmod p$. Then the zeta function of ~~Xto~~ X_{t_0} can be expressed as

$$Z(X_{t_0}, T, \mathbb{F}_p) = \frac{(1 - \pi_{t_0} T)(1 - \frac{p}{\pi_{t_0}} T)}{(1-T)(1-pT)}, \quad \pi_{t_0} = \left.\frac{\phi(t)}{\phi(t^p)}\right|_{t=\hat{t}_0}$$

Teichmüller lift

So we may compute π_{t_0} by computing $(*)$ mod p^{s+1}.

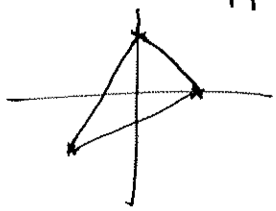
Laurent polynomials

$$X^{\underline{a}} = X_1^{a_1} \cdots X_n^{a_n}, \quad \underline{a} = (a_1, \dots, a_n) \in \mathbb{Z}^n$$

$$f = \sum c_{\underline{a}} X^{\underline{a}} \in \mathbb{Z}[X_1, X_1^{-1}, \dots, X_n, X_n^{-1}]$$

$$\operatorname{supp}(f) = \{\underline{a} \in \mathbb{Z}^n \mid c_{\underline{a}} \neq 0\}$$

Newton polyhedron $\Delta(f)$ = convex hull of $\operatorname{supp}(f)$

[eg. $f = X + Y + \frac{1}{XY}$, $\Delta(f) =$ ]

Data of $\Delta(f)$ can be encoded in a matrix A: if f has m monomials, A is an $n \times m$-matrix with columns $\underline{a}_j$, $1 \le j \le m$, $\underline{a}_j$ are the exponents of f.

[eg. $A = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix}$]

$\Delta(f)$ is the image of the standard simplex Δ_m under the map $\mathbb{R}^m \xrightarrow{A} \mathbb{R}^n$.

Lemma: Let Δ be an integral polyhedron with 0 as a unique interior lattice point. Then for all non-negative integral vectors $(l_1, \dots, l_m) \in \mathbb{Z}^m$ s.t. $\sum_{j=1}^m a_{ij} l_j \neq 0$ for some i, one has

$$g := \gcd\Big(\sum_{j=1}^m a_{ij} l_j\Big) \le \sum_{j=1}^m l_j$$

Proof: Assume that there exists a vector $(l_1, \dots, l_m)$ s.t. $\sum_{j=1}^m a_{ij} l_j \neq 0$ for some i, and $g > \sum_{j=1}^m l_j$, then

$$\underline{v} = \frac{1}{g}\left(\underline{a}_1 l_1 + \dots + \underline{a}_m l_m\right) = \frac{1}{g}\begin{pmatrix} \sum a_{1j} l_j \\ \vdots \\ \sum a_{nj} l_j \end{pmatrix} \text{ is s.t. } \sum \frac{l_i}{g} < 1 \quad \therefore \underset{0}{\overset{\neq}{\underline{v}}} \in \overset{\circ}{\Delta}(f)$$

$\#$

The fundamental period

Let $[f]_0$ denote the constant term of the Laurent polynomial f. The fundamental period $\phi(t)$ of f is the power series
$$\phi(t) = \sum_{n=0}^{\infty} [f^n]_0 t^n .$$

Theorem: Let $f \in \mathbb{Z}[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$ and assume that $\Delta(f)$ has 0 as a unique interior lattice point. Then the coefficients $a(n) = [f^n]_0$ satisfy D3 for all p.

Proof for (mod p): want $a(n_0 + n_1 p) \equiv a(n_0) a(n_1) \bmod p$.

i.e. want $[f^{n_0} f^{n_1 p}]_0 \equiv [f^{n_0}]_0 [f^{n_1}]_0 \bmod p$ $(**)$

Have $f^{n_1 p}(X) \equiv f^{n_1}(X^p) \bmod p$, $(n_1 < p)$ so $(**)$ is implied by $[f^{n_0}(X) f^{n_1}(X^p)]_0 \equiv [f^{n_0}]_0 [f^{n_1}]_0$. $(***)$

Assume there is a monomial M in f^{n_0} which is a monomial in X^p. $M = X^{l_1 \underline{a}_1} \cdots X^{l_m \underline{a}_m}$, $l_1 + \dots + l_m = n_0 < p$

then M monomial in $X^p \Rightarrow p \mid \sum_{j=1}^{m} a_{ij} l_j$ for each i

$\Rightarrow p \mid \gcd$. $\not\times$, since by the lemma, M is constant, so $(***)$ can only occur on constants.

(Proof for higher congruences similar)