Ilya Shapiro 18/03/10 11:00

Frobenius maps for quintic threefolds

Original motivation: integrality results

Reduces to a vanishing of a matrix element of Fr.
(use motives)

Goal: See what you can do by hand for the quintic

Setup: V_λ zeros of $\varphi_\lambda = \lambda(x_0^5 + \dots + x_4^5) + x_0 \dots x_4$ in $\mathbb{P}^4$

$\lambda \neq 0$ small, $H^3(V_\lambda)$

i) fit together into a flat bundle w/ G-M connection ∇_λ, $\delta = \lambda \nabla_\lambda$

ii) Each fibre has $(\, , \,)$

iii) $Fr^* : H^3(V_{\lambda^p}) \to H^3(V_\lambda)$

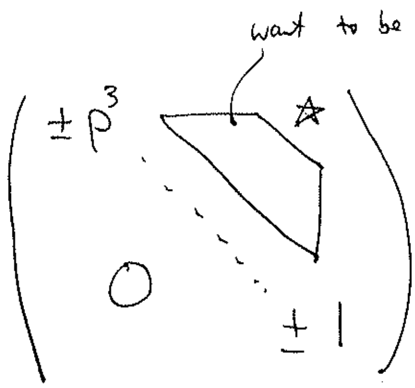
iv) ω section of $H^3(V_\lambda)$ given by the CY-form

$H^3(V_\lambda)^{inv}$ extended to $\lambda = 0$ by taking $\mathcal{H} = \mathrm{span}(\omega, \delta\omega, \delta^2\omega, \delta^3\omega)$

Precise goal: Compute $Fr^*|_0$ in $\delta^{i \leq 3}\omega$ basis

Observation: $(\delta^{i<3}\omega)_0 = 0$

From general considerations:



Prove: • $a_{12} = 0$ sufficient $\Rightarrow$ all but $\star$ are 0.

• Look at $\star$

Use Dwork cohomology:

$$DR^6\, \mathbb{C}_p^\dagger \langle \overbrace{x_0, \dots, x_4}^{\underline{x}}, t^{-5} \rangle_0 [[\lambda^{\pm 1}]]\, e^{\pi t \varphi_\lambda} \qquad (\pi^{p-1} = -p)$$

$\omega = dx\,dt \qquad Fr(\omega(x,t,\lambda)) = e^{\pi(t^p \varphi_{\lambda^p}(x^p) - t\varphi_\lambda)}\, \omega(x^p, t^p, \lambda^p)$

$$\nabla_\lambda = \partial_\lambda + (\partial_\lambda \pi t \varphi_\lambda)$$

$\mathcal{H} = \mathbb{C}_p[[\lambda]]$-span of $(xt)^{i \leq 3} dx\,dt$

$$e^{\pi(-)} = A(\lambda t x_0^5) \dots A(\lambda t x_4^5) A(xt)$$

where $A(z) = e^{\pi(z^p - z)}$

$$\omega_I = \omega_{ijkmns} := (\lambda t x_0^5)^i \dots (\lambda t x_4^5)^n (tx)^s$$

$\Rightarrow$ $\delta \omega_I = -(s+1)\omega_I - \pi \omega_{I+1_s}$

& $\delta \omega_I = si\,\omega_I + s\pi \omega_{I+1_i}$

$$c_I^\alpha = \frac{(-1)^I \pi^I}{Y} (\omega_I, \delta^\alpha \omega)_0, \qquad Y = (\omega, \delta^3 \omega)_0$$

$$D = \sum_{i=0}^\infty \partial_x^i, \qquad {}^\alpha D_x = (D\tfrac{1}{x})^\alpha D, \qquad {}^\alpha D^\beta = [{}^\alpha D_x x^\beta]_0$$

${}^\alpha D^\beta$ satisfy similar relations to c_I^α.

Rmk: $g(x) = \sum a_\beta x^\beta \Rightarrow \sum a_\beta {}^\alpha D^{\beta+k} = [{}^\alpha D_x x^k g(x)]_0$

$${}^\alpha Q^s \sim {}^\alpha D^s$$
$${}^\alpha p^i \sim {}^\alpha D^{i-1}$$

$$C_{I}^{\alpha} = \sum_{\gamma+\beta_1+\dots+\beta_5=\chi} {}^{\gamma}Q^{s}\, {}^{\beta_1}p^{i} \dots {}^{\beta_5}p^{m}, \quad \chi = 3-\alpha-\#\text{ of non-0s among } i,\dots,n \text{ in } I$$

$$f(x) = \exp\left(\frac{x^p}{p} + x\right) =: \sum B_i x^i \qquad [A(z) = f(-\pi z)]$$

$$(\mathrm{Fr}\,\omega, \delta^{\alpha}\omega)_0 = -p^5 Y \sum B_i \dots B_s C^{\alpha}_{i\dots m(s+p-1)}$$

$$x^{p-1} f(x) = \partial_x f(x) - f(x)$$

$$\Rightarrow \boxed{{}^{s}D_x\, x^{p-1} f(x) = -\,{}^{s-1}D_x \frac{1}{x} f(x), \quad s \geq 1}$$

$$(\mathrm{Fr}\,\omega, \delta^{3}\omega)_0 = p^5 Y \quad \text{(1-line)}$$

$$(\mathrm{Fr}\,\omega, \delta^{2}\omega)_0 = 0 \quad \text{(2-lines)} \quad \Rightarrow a_{12} = 0$$

$$\Rightarrow (\mathrm{Fr}\,\omega, \delta\omega)_0 = 0$$

obtain $\boxed{\left[{}^{1}D_x \frac{1}{x} f(x)\right]_0 = \frac{\left[{}^{0}D_x \frac{1}{x} f\right]_0^2}{2}}$ (6-lines)

$$(\mathrm{Fr}\,\omega, \omega)_0 = 0 \overset{\text{(9-lines)}}{\iff} \left[{}^{2}D_x \frac{1}{x} f\right]_0 = \frac{\left[{}^{0}D_x \frac{1}{x} f\right]_0^3}{3!}$$

but it is $\neq 0$;

$\boxed{a^{\star}_{14}{}'' = \frac{2^3(p^3-1)}{5^2} \zeta_p(3)}$ (from numerical calculations)