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Logarithmic Growth and Frobenius Slopes

1. log-growth

K complete discrete valuation field, $|p| = p^{-1}$, σ: p-Frobenius

$L \in K[x, \frac{d}{dx}]$, $\alpha \in K$ nonsingular pt. w.r.t. L

$$Sol(L, \alpha) = \left\{ f = \sum_{n=0}^{\infty} a_n (x-\alpha)^n \;\middle|\; L(f) = 0 \right\}$$

$\dim = \mathrm{rk}\, L$

L geometric (with Frobenius structure)

$\Rightarrow |a_n| \eta^n \to 0 \quad (\forall 0 < \eta < 1)$

$\limsup |a_n| = \infty$ in general

Dwork: $\gamma \geq 0$, $\sum a_n (x-t)^n$ is of log-growth $\leq \gamma$

$\Leftrightarrow |a_n| = O(n^\gamma)$

"bounded" $\Leftrightarrow \sup |a_n| < \infty \Leftrightarrow$ l-g $= 0$

eg. 1) $L = x\frac{d}{dx} - \underbrace{\begin{pmatrix} 0 & 1 & & \\ & & \ddots & \\ & & & 1 \\ & & & 0 \end{pmatrix}}_{=G}$ $\quad$ rk $= r$

$\alpha \in K$, $\bar{\alpha} \neq 0, \infty$

solutions $\quad x\frac{d}{dx} Y = YG \qquad Y = \begin{pmatrix} 1 & \ell & \cdots & \frac{1}{(r-1)!}\ell^{r-1} \\ & \ddots & \ddots & \vdots \\ & & \ddots & \ell \\ 0 & & & 1 \end{pmatrix}$

where $\ell = \log\left(1 + \frac{x-\alpha}{\alpha}\right) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(x-\alpha)^n}{n\,\alpha^n}$

so l-g of $\ell = 1$ & l-g of $\ell^j = j$

So have a basis of solns of l-g's $0, 1, \ldots, r-1$ (rows in Y).

2) $L = x^2(1-x)\left(\frac{d}{dx}\right)^2 - x(1-2x)\frac{d}{dx} - \frac{x}{4}$

$${}_2F_1\left(\begin{matrix}\frac{1}{2}, \frac{1}{2}\\ 1\end{matrix}; x\right)$$

$H^1\left(LF/\mathbb{P}^1\setminus\{0,1,\infty\}\right)$ Gauss-Manin

$\underline{\alpha = \frac{1}{2}}$, $j = 1728$ $\mathbb{Q}(\sqrt{-1})$

So if $f = \sum a_n (x-\frac{1}{2})^n$ is a solution, then

$$a_{n+2} = \frac{(2n+1)^2}{(n+1)(n+2)} a_n .$$

l-g's:

		$p \equiv 1 \pmod 4$	$p \equiv 3 \pmod 4$
f_0	$a_0 = 1$, $a_1 = 0$	0	$\frac{1}{2}$
f_1	$a_0 = 0$, $a_1 = 1$	$1^{(*)}$	$\frac{1}{2}$
		ord. red?	s.s. red?

(*) $a_{p^r} = \dfrac{(2p^r-3)^2(2p^r-7)^2\ldots 3^2}{p^r!}$

$\operatorname{ord}_p(\text{denom.}) = \dfrac{p^r-1}{p-1}$

$1 \le p^n(4m-1) \le 2p^r$

$1 \le m \le \dfrac{p^{r-h}-1}{2} + \dfrac{1}{4}$

$\therefore \operatorname{ord}_p(\text{numer.}) = 2 \times \sum_{h=1}^{r} \dfrac{p^{r-h}-1}{2} = \dfrac{p^r-1}{p-1} - r$

$\therefore \operatorname{ord}_p(a_{p^r}) = -r \Rightarrow$ l-g of $f_1 \geq 1$

sim. $\operatorname{ord}_p(a_n) \lesssim$ "$\log n$" $\Rightarrow$ l-g of $f_1 = 1$

Dwork: HGE of order 2

nontrivial case: l-g = Frob. slope

$M \qquad M^\vee(-1)$

trivial case:

on (α), $M = \underset{\varphi\text{-module}/K}{L} \otimes K[\![x-\alpha]\!]_0 \rightsquigarrow$ l-g $= 0$

Rmk 1) lg at generic pt. $x=t$, $|t|=1$ (t transcendental / residue field)

2) Kedlaya: regular sing. case, "l-g of "log" $= 1$

3) $\exists$ example s.t. lg $\notin \mathbb{Q}$ without Frob

4) Dwork's conjecture on log-growth

Newton polygon of lg at sp. pt. $\geq$ NP of lg at gen. pt.

Conjecture: $\lambda \geq 0$, $K[\![x]\!]_\lambda = \left\{ \sum_{n=0}^{\infty} a_n x^n \;\middle|\; |a_n| = O(n^\lambda) \right\}$

$S = K[\![x]\!]_0 = \mathcal{O}_K[\![x]\!] \otimes K \circlearrowleft \sigma$: ext. of Frobenius

$\therefore K[\![x]\!]_\lambda \circlearrowleft \sigma$ Frobenius

$\mathcal{E} = \left\{ \sum_{n=-\infty}^{\infty} a_n x^n \;\middle|\; a_n \in K, \sup|a_n| < \infty, a_n \to 0 \ (n \to -\infty) \right\}$ $\circlearrowleft \sigma$

comp. disc. val. field $/K$

$= \mathrm{Frac}(S)^\wedge$

M (φ, ∇)-module $/S$

$\Leftrightarrow M \begin{cases} \text{is a free } S\text{-module of finite rank} \\ \text{has } \nabla : M \to M \otimes \Omega^1_S \ K\text{-connection} \\ \text{has Frobenius } \varphi : \sigma^* M \xrightarrow{\sim} M \text{ horizontal} \end{cases}$

M (φ, ∇)-mod. / S

$\rightsquigarrow M_\eta = M \otimes \mathcal{E}$

- $S_\beta(M_\eta)$: Frob. slope filtration

  S_β / S_{β^-} : pure of slope β, where $S_{\beta^-} = \bigcup_{\gamma < \beta} S_\gamma$

- $(M_\eta)^\lambda$: log-growth fil. $(\lambda \geq 0)$ (Robba)

  $M_\eta / (M_\eta)^\lambda$: max. quotient s.t. all sol$^{\text{n}}$s of l-g $\leq \lambda$ of $M_{\eta_{K_\rho}}$ factor through M_η / M_η^λ (at gen. pt.)

- $M_\eta^\lambda \neq M_\eta$ $\forall \lambda \geq 0$
- $M_\eta^{\mathrm{rk} M - 1} = 0$

$\xrightarrow{\text{spec. fibre.}} V(M) = M \otimes K$, φ-mod.

$S_\beta V(M)$: Frob. slope fil.

$$Sol_\lambda(M) = \mathrm{Hom}_{S[\frac{d}{dx}]}(M, K[[x]]_\lambda)$$

$$Sol_\lambda(M) = Sol(M) \quad \text{if} \quad \lambda \geq \mathrm{rk} M - 1$$

Theorem A:

M : (φ,∇)-module$/S$

β_{max} : ~~the~~ highest Frob-slope of M_η

$\Rightarrow \dim Sol_\lambda(M) + \dim S_{(\beta_{max}-\lambda)^{-}}(V(M)) \geq rk\, M$

Definition: M (φ,∇)-module$/S$

M PBQ (of pure bounded quotient) $\Leftrightarrow$ M_η / M_η^0 has pure Frob. slope

Conjecture: M PBQ $\Rightarrow$ "=" in Theorem A.

Rmk: 1) M_η bounded (φ,∇)-mod.$/S$

$\Rightarrow M_\eta \cong \bigoplus_\beta S_\beta / S_{\beta^-}$

2) $\exists\, P \subset M$, $P \neq 0$ s.t. i) P is PBQ

ii) $\beta_{max}(P_\eta) = \beta_{max}(M_\eta)$

eg. 1) $M = H^1(LF/\mathbb{P}^1\setminus\{0,1,\infty\}) \otimes k[\![x-\alpha]\!]_0$, $\bar{\alpha} \neq 0,1,\infty$

$\Rightarrow M$ is PBQ since $\dim M_\eta / M_\eta^0 = 1$

2) $M = M_0 \otimes S$ $\quad M_0$ φ-mod.$/S$

$\Rightarrow M$ is PBQ $\Leftrightarrow$ M_0 is pure

Theorem:

Conj. is true for rk ≤ 2

Defn: M is HBQ (horizontal of b.q.) $\iff$ $\exists\, N \subset M \,/\, S$ s.t.

$$\left(M/N\right)_\eta \cong M_\eta / M_\eta^0$$

eg. i) $M = H^1(LF) \otimes K[\![x-\alpha]\!]_0$

M HBQ $\iff$ λ ordinary

Thm: M PBQ & HBQ $\Rightarrow$ Conj. is true (ie. "=" in Thm A)

To prove: **Lemma**: $p^\beta \in K$ $|p^\beta| = p^{-\beta}$, $\sigma(p^\beta) = p^\beta$

$f = \sum a_n x^n \in K[\![x]\!]_\lambda$.

$p^\beta y - \sigma(y) = f$, $y \in K[\![x]\!]$

Then i) $y \in K[\![x]\!]_\lambda$

ii) If $a_n = 0$ for $\beta | n \Rightarrow y$ is exactly of l-g?