Ordinary crystals with logarithmic poles

1. The Definitions

$A = W[[\lambda]]$, W = Witt ring of $k = \bar{k}$, $\mathrm{char}(k) = p > 0$

"Put log structure along $\lambda = 0$"

$\Omega^1 = A\frac{d\lambda}{\lambda}$

Lifting φ of Frob : $\begin{matrix} k \to k \\ x \mapsto x^p \end{matrix}$ to A

$\Rightarrow \varphi : \lambda \mapsto \lambda^p f$, $f \equiv 1\ (p)$

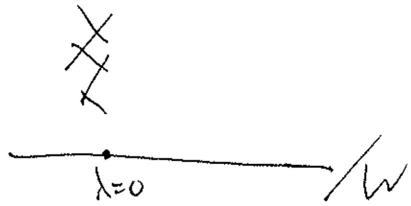
Defn: 1) Crystal (H, ∇) : • H a free A-module
• $\nabla : H \to \Omega^1_A \otimes H$ "quasi-unipotent"
$\lambda^n \left(\frac{d}{d\lambda}\right)^n \xrightarrow{p\text{-adically}} 0$ as $n \to \infty$

2) F-crystal (H, ∇, F) : • (H, ∇) crystal
• F Frobenius structure : for any ϕ lifting of Frob,

$$\begin{tikzcd} \phi^* H \arrow[r, "F(\phi)"] \arrow[d, "\phi^*\nabla"'] \arrow[dr, phantom, "\circlearrowleft"] & H \arrow[d, "\nabla"] \\ \Omega^1 \otimes \phi^* H \arrow[r, "\mathrm{id} \otimes F(\phi)"] & \Omega^1 \otimes H \end{tikzcd}$$

, $F(\phi) \otimes \mathbb{Q}$ is bijective.

3) Hodge F-crystal $(H, \nabla, F, \mathrm{Fil}^\bullet)$:
• $\nabla \mathrm{Fil}^{i+1} \subset \Omega^1 \otimes \mathrm{Fil}^i$
• $F(\phi)\phi^* \mathrm{Fil}^i \subset p^i H$

eg. $H^i_{\text{log-crys}}\Bigg($  $\Bigg)$

The Newton polygon at $\lambda = 0$

$$(H, \nabla, F, \mathrm{Fil}^\bullet) \otimes_A W$$

$$\Rightarrow \underbrace{(e^*H}_{\text{free } W\text{-module}}, N, F, \mathrm{Fil}^\bullet), \quad \boxed{pFN = NF}$$

<u>Defn</u>: A Hodge F-crystal is ordinary if Hodge = Newton ($\lambda = 0$)

2. The results

Suppose H is ordinary.

$$H = \mathrm{Fil}^0 \supset \mathrm{Fil}^1 \supset \mathrm{Fil}^2 = 0$$

rk $\quad 2 \quad\quad 1$

Then $H = U_0 \oplus \mathrm{Fil}^1$, $U_0 = \langle u_0 \rangle$ (sub F-crystal), $\mathrm{Fil}^1 = \langle u_1 \rangle$ s.t.

$$\nabla u_0 = 0$$
$$\nabla u_1 = \text{"}d\log q\text{"} \otimes u_0$$
$$F(\phi) u_0 = u_0$$
$$F(\phi)\phi^* u_1 = p u_1 + p? u_0$$

where $d\log q = (a_0 + a_1\lambda + \dots)\dfrac{d\lambda}{\lambda}$

$\Rightarrow \log q = (a_0 \log\lambda + a_1\lambda + \dots) + c'$

Can normalise c' s.t. $\phi^*\log q - p\log q - p? = 0 \quad \forall \phi$

$(p>2) \Rightarrow \quad q = c\lambda^{a_0}(1 + b_1\lambda + \dots)$

$c = \exp(c')$ is well-defined since $p \mid c'$

One can pick $a_0 = 1$, if

$$\begin{tikzcd} \mathrm{Fil}^1 \arrow[r, "\delta = \lambda\partial_\lambda"] \arrow[dr] & H \arrow[d] \\ & H/\mathrm{Fil}^1 \end{tikzcd}$$

Application I

Calabi-Yau $/\mathbb{Q}$

$\operatorname{rk} H^3 = 4$ $\qquad \exists \left.\begin{array}{l} \nabla \tilde{u}_0 = 0 \\ \nabla \tilde{u}_1 = d\log \tilde{q} \otimes \tilde{u}_0 \end{array}\right\} /\mathbb{C}$

$\quad\; \| $

$H^3(\Phi_\lambda)^{\mathrm{inv}}$

if we normalise s.t. $\tilde{q} = \lambda(1 + \dots)$

$$\Rightarrow c\tilde{q} = q$$

$$\Rightarrow \tilde{q} \in \mathbb{Z}_p$$

$wt = 3$ $\qquad H = \langle \underbrace{u_0, u_1}_{\text{as before}}, u_2, u_3 \rangle \;:\; \langle u_0, u_3 \rangle = 1 = \langle u_2, u_3 \rangle$

$$\nabla \underline{u} = \begin{pmatrix} 0 & & & \\ 0 & d\log q & & \\ & 0 & \kappa \cdot d\log q & \\ & & 0 & -d\log q \end{pmatrix} \underline{u}$$

$\phi : \underline{q} \mapsto q^p$ $\qquad F(\phi)\,\phi^* \underline{u} = \begin{pmatrix} 1 & & & \\ 0 & p & & \\ p^2 a & p^2 b & p^2 & \\ p^3 c & p^3 a & & p^3 \end{pmatrix} \underline{u}$

$$-\tfrac{1}{2} c \xmapsto{\delta = \lambda \frac{d}{d\lambda}} a \xmapsto{\delta} b \xmapsto{\delta} \phi(K) - K$$

[KSV] $\qquad K = K(0) + \sum_{n \geq 1} \frac{n^3 q^n}{1 - q^n} K(n)$

$$\Rightarrow K(n) \in W$$

$\sim$

3. The periods

$$\begin{tikzcd} A \arrow[rr, dashed] \arrow[dr] & & W \arrow[dl] \\ & k & \end{tikzcd}$$

need log-structure :

$$\begin{tikzcd} (A, \mathcal{L}) \arrow[rr, dashed, "f = (f_0, f_1)"] \arrow[dr] & & (W, \mathbb{N} \oplus W^\times) \arrow[dl] \\ & (k, \mathbb{N} \oplus k^\times) & \end{tikzcd}$$

$0 \leftarrow (1,1)$

$H = \langle u_0, u_1 \rangle$ $\qquad \nabla u_1 = d\log q \otimes u_0$

$\qquad\qquad\qquad\quad F u_1 = p u$ $\qquad \phi : q \mapsto q^p$

$f_0(\lambda) = 0$

$f_1(1,1) = (1,\beta)$ with $\beta \in 1 + pW$

$\mathrm{Fil}^1 e^* H = \langle u_1 + \log\beta \cdot u_0 \rangle$