REVIEW OF TOPOLOGICAL FIELD THEORY AND HOMOLOGICAL MIRROR SYMMETRY

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ABSTRACT. We review Witten's topological field theory and obtain the A-model and the B-model. We discuss that Gromov-Witten invariants arise because of a fixed-point theorem for BRST operators. By considering more general observables, we motivate the deformation of topological field theory and introduce branes to the theory. Branes demand new mathematical tools such as sheaves and categories. We finally justify and present the category of A-branes and B-branes, and state the homological mirror symmetry conjecture. The first half of this article is primarily based on [Wit92], and the second half on [Asp04].

1. INTRODUCTION

In the low energy limit, string theory reduces to superconformal field theory on a Riemann surface Σ. The motivation for mirror symmetry occurred because when reducing string theory on to a superconformal field theory, it was a matter of pure convention to decide which parameter corresponds to the complex structure and which corresponds to the Kähler class of the underlying manifold. This means that there are two Calabi-Yau manifolds X and Y called *mirror pairs*, that produces the same physical action in the effective field theory limit.

Witten constructed two dimensional topological field theories on the mirror pairs, by twisting the transformation laws of the fields in the original physical Lagrangian. By examining the properties of the two mirror models, the A-model on Y and the B-model on X, we see that the supersymmetry generator changes to a nilpotent operator that acts as the BRST operator in the model. Furthermore, the BRST cohomology of the A and B model corresponds to the de Rham cohomology of Y, and the Dolbeault cohomology of X, respectively. By looking at the observables of the two models, we can make contact with the real world. Observables are, by definition, BRST invariant operators. By considering the path integral of the observable, we can show that only the fixed points under BRST variation gives nonzero contribution to the expectation values. This will explain the fundamental reason why the mirror map is related to the Gromov-Witten invariants.

We can generalize this approach further by including open strings to the theory, which inevitably introduces D-branes to it. In describing D-branes, it is necessary to introduce new mathematical tools, such as category theory and sheaf theory. For the A-model, the category of D-branes form the Fukaya category of Y, that is $\mathcal{F}(Y)$. For the B-model the category of D-branes form the derived category $\mathbf{D}(X)$. If we consider stability issues of the D-branes, which relates to the BPS states of the physical theory, we can finally state the homological mirror symmetry conjecture in terms of the triangulated categories.

In section 2, we will review Witten's topological field theory [Wit92] and discuss about the properties of the A-model and the B-model. In section 3, we will outline the category of D-branes. The derivation of the category of A-branes is highly technical. We omit most of its technical details, and only sketch the steps in obtaining such category. The derivation of the category of B-branes is simpler and we will give a relatively detailed description of the procedure in constructing it. In section 4, we discuss the stability of the D-branes, and finally state the homological mirror symmetry conjecture.

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2. Topological Field Theory

In this section we review Witten's 2-dimensional closed string topological field theory with $N = (2, 2)$ supersymmetry. The two topological field theories, called A-theory and B-theory, are constructed from the physical field theory by twisting the fields. After the twist, one can show that the supersymmetry charge Q is nilpotent and can be thought as a BRST operator. Surprisingly, BRST cohomology of the quantum fields and the cohomology of the target manifold are homomorphic, thus called topological field theory. BRST cohomology also plays an important role on the Gromov-Witten invariants via the analog of the fixed point theorem. In the end of this section, we motivate the study of D-branes. This section is based on Witten's paper [Wit92].

2.1. **Physical model.** The nonlinear sigma model is consisted of a map ϕ from a Riemann surface Σ to a target space X, sections of the canonical line bundle K, and the pull-back of the complexified tangent bundle of X. In this article, we assume the target space is Kähler and furthermore a Calabi-Yau manifold. The action of the nonlinear sigma model with $N = 2$ supersymmetry is given by,

$$S = 2t \int d^2z \Big(\frac{1}{2} g_{IJ} \partial_z \phi^I \partial_{\bar{z}} \phi^J + i B_{i\bar{j}} (\partial_z \phi^i \partial_{\bar{z}} \phi^{\bar{j}} - \partial_{\bar{z}} \phi^i \partial_z \phi^{\bar{j}}) + i\psi_-^{\bar{i}} D_z \psi_-^i g_{\bar{i}i} + i\psi_+^{\bar{i}} D_{\bar{z}} \psi_+^i g_{\bar{i}i} + R_{i\bar{i}j\bar{j}} \psi_+^i \psi_+^{\bar{i}} \psi_-^j \psi_-^{\bar{j}} \Big).$$

Where $z, \bar{z}$ are the coordinates on the Riemann surface, ψ_+ and ψ_- are the sections of $K^{1/2} \otimes \phi^*(TX)$ and $\overline{K}^{1/2} \otimes \phi^*(TX)$ respectively. We use the I, J index for the real coordinates of the tangent space, and $i, j, \bar{i}, \bar{j}$ the holomorphic and antiholomorphic parts of the complexified coordinates respectively. In terms of the infinitesimal fermionic parameters $\alpha_-, \widetilde{\alpha}_-$, which are holomorphic sections of $K^{-1/2}$, and $\alpha_+, \widetilde{\alpha}_+$, which are the antiholomorphic sections of $\overline{K}^{-1/2}$, the transformation laws can be written as,

$$\begin{aligned}
\delta\phi^i &= i\alpha_-\psi_+^i + i\alpha_+\psi_-^i \\
\delta\phi^{\bar{i}} &= i\widetilde{\alpha}_-\psi_+^{\bar{i}} + i\widetilde{\alpha}_+\psi_-^{\bar{i}} \\
\delta\psi_+^i &= -\widetilde{\alpha}_-\partial_z\phi^i - i\alpha_+\psi_-^j\Gamma^i_{jm}\psi_+^m \\
\delta\psi_+^{\bar{i}} &= -\alpha_-\partial_z\phi^{\bar{i}} - i\widetilde{\alpha}_+\psi_-^{\bar{j}}\Gamma^{\bar{i}}_{\bar{j}\bar{m}}\psi_+^{\bar{m}} \\
\delta\psi_-^i &= -\widetilde{\alpha}_+\partial_{\bar{z}}\phi^i - i\alpha_-\psi_+^j\Gamma^i_{jm}\psi_-^m \\
\delta\psi_-^{\bar{i}} &= -\alpha_+\partial_{\bar{z}}\phi^{\bar{i}} - i\widetilde{\alpha}_-\psi_+^{\bar{j}}\Gamma^{\bar{i}}_{\bar{j}\bar{m}}\psi_+^{\bar{m}}.
\end{aligned}$$

The topological field theory is constructed by twisting the above transformation laws, by changing the sections which the fields corresponds to. We define the twists in the following table,

Twists			
	sections of	+ twist	- twist
ψ_+^i	$K^{1/2}\otimes\phi^*(T^{1,0}X)$	$\phi^*(T^{1,0}X)$	$K\otimes\phi^*(T^{1,0}X)$
$\psi_+^{\bar{i}}$	$K^{1/2}\otimes\phi^*(T^{0,1}X)$	$K\otimes\phi^*(T^{0,1}X)$	$\phi^*(T^{0,1}X)$
ψ_-^i	$\overline{K}^{1/2}\otimes\phi^*(T^{1,0}X)$	$\phi^*(T^{1,0}X)$	$\overline{K}\otimes\phi^*(T^{1,0}X)$
$\psi_-^{\bar{i}}$	$\overline{K}^{1/2}\otimes\phi^*(T^{0,1}X)$	$\overline{K}\otimes\phi^*(T^{0,1}X)$	$\phi^*(T^{0,1}X)$

Using these twists, we can construct the A-model and B-model. We will concentrate on describing the transformation laws and the BRST cohomology for each model, since these will be the relevant properties for our discussion. For further details consult [Wit92].

2.2. **A-model.** The A-model is obtained by taking a + twist of ψ_+ and a - twist of ψ_-. Furthermore, we take $\alpha_+ = \widetilde{\alpha}_- = 0$, and set α_- and $\widetilde{\alpha}_+$ to constants, which we call α and $\widetilde{\alpha}$. The transformation law for the A-model is,

$$\begin{aligned}
\delta\phi^i &= i\alpha\chi^i \\
\delta\phi^{\bar{i}} &= i\widetilde{\alpha}\chi^{\bar{i}} \\
\delta\chi^i &= \delta\chi^{\bar{i}} = 0 \\
\delta\psi_z^{\bar{i}} &= -\alpha\partial_z\phi^{\bar{i}} - i\alpha\chi^{\bar{j}}\Gamma^{\bar{i}}_{\bar{j}\bar{m}}\psi_z^{\bar{m}} \\
\delta\psi_{\bar{z}}^i &= -\widetilde{\alpha}\partial_{\bar{z}}\phi^i - i\alpha\chi^j\Gamma^i_{jm}\psi_{\bar{z}}^m,
\end{aligned} \tag{1}$$

where we have introduced χ as a section of $\phi^*(TX)$, by defining $\chi^i = \psi_+^i$ and $\chi^{\bar{i}} = \psi_-^{\bar{i}}$. As in the nonabelian gauge theories, we can use the standard procedure and construct the BRST operator Q, and express the transformations in terms of the BRST variation given by $\delta\varphi = -i\alpha\{Q,\varphi\}$ for any field φ.

One interesting aspect of the A-model is that the observables are related to the de Rham cohomology. Let $W = W_{I_1I_2\ldots I_n}(\phi)d\phi^{I_1}d\phi^{I_2}\cdots d\phi^{I_n}$ be a n-form on X. We relate the following local operator in the field theory to W,

$$\mathcal{O}_W(P) = W_{I_1I_2\ldots I_n}\chi^{I_1}\cdots\chi^{I_n}(P)$$

where the ghost number of $\mathcal{O}_W$ is n. Following the standard BRST variation calculations we obtain,

$$\{Q, \mathcal{O}_W\} = -\mathcal{O}_{dW},$$

with d the exterior derivative on W. This shows that the BRST cohomology of the quantum field theory is homomorphic to the de Rham cohomology of X.

2.3. **B-model.** The B-model is obtained by making - twists on both ψ_+ and ψ_-. We introduce η, θ, and ρ that will reduce the complexity of the equations, as the following,

$$\begin{aligned} \eta &= \psi_+^{\bar{i}} + \psi_-^{\bar{i}} \\ \theta &= g_{i\bar{i}}(\psi_+^{\bar{i}} - \psi_-^{\bar{i}}) \\ \rho &= (\rho_z^i, \rho_{\bar{z}}^i) = (\psi_+^i, \psi_-^i). \end{aligned}$$

For the transformation laws, we set $\alpha_\pm = 0$, and $\widetilde{\alpha}_+ = \widetilde{\alpha}_- = \alpha$. The transformation laws are then,

$$\begin{aligned} \delta\phi^i &= 0 \\ \delta\phi^{\bar{i}} &= i\alpha\eta^{\bar{i}} \\ \delta\eta^{\bar{i}} &= \delta\theta_i = 0 \\ \delta\rho^i &= -\alpha d\phi^i. \end{aligned} \tag{2}$$

As in the A-model case, we can construct a BRST operator which is a nilpotent.

Similarly to the A-model, we can associate the observables to differential forms. For the B-model, a $(0,p)$ form V on X with values in $\wedge^q T^{1,0}X$, can be explicitly written as,

$$V = d\overline{z}^{i_1} d\overline{z}^{i_2} \cdots d\overline{z}^{i_p} V_{\bar{i}_1\bar{i}_2\cdots\bar{i}_p}{}^{j_1 j_2 \cdots j_q} \frac{\partial}{\partial z_{j_1}} \cdots \frac{\partial}{\partial z_{j_q}}.$$

The sheaf cohomology group $H^p(X, \wedge^q T^{1,0}X)$ consists of solutions of $\overline{\partial} V = 0$ modulo the exact form of $\overline{\partial}$. We relate V to a quantum field theory operator $\mathcal{O}_V$, which is defined by

$$\mathcal{O}_V = \eta^{\bar{i}_1} \cdots \eta^{\bar{i}_p} V_{\bar{i}_1\ldots\bar{i}_p}{}^{j_1\ldots j_q} \psi_{j_1} \cdots \psi_{j_q}$$

For the B-model, we can derive,

$$\{Q, \mathcal{O}_V\} = -\mathcal{O}_{\overline{\partial}V},$$

which means the BRST cohomology is given by the sheaf cohomology.

2.4. **The Fixed Point Theorem.** In this section, we will explore why calculations on the A-model reduce to integrals over moduli spaces of holomorphic curves, while calculations on B-model reduce to integrals over spaces of constant maps.

Consider a quantum field theory, with a path integral over $\mathcal{E}$. Suppose the theory has a symmetry group F which acts *freely* on $\mathcal{E}$. When we consider the path integral of F-invariant observables $\mathcal{O}$, we obtain,

$$\int_{\mathcal{E}} e^{-S}\mathcal{O} = \mathrm{vol}(F) \int_{\mathcal{E}/F} e^{-S}\mathcal{O}. \tag{3}$$

Now lets consider the symmetry group G generated by the BRST operator Q. The volume of G is zero, because of the integral of the Grassmannian θ is defined by,

$$\int d\theta 1 = 0.$$

This tells us that if the BRST symmetry group G acts freely on $\mathcal{E}$, by (3), the expectation value of any BRST invariant operator $\mathcal{O}$ will be zero. Therefore the only contributions to the nonzero expectation value of $\mathcal{O}$ comes from the fixed points of G. In other words, we are interested in the subspace $\mathcal{E}_0 \subset \mathcal{E}$, that the BRST transformation laws become zero.

For the A-model, by requiring the transformations in (1) to be zero, we obtain the equations for holomorphic curves. This can be seen by the following reasoning. Requiring, $\delta\phi^I = 0$ gives $\chi^I = 0$, and $\delta\psi = 0$ gives

$$\partial_{\bar{z}}\phi^i = \partial_z\phi^{\bar{i}} = 0.$$

This is the equations of holomorphic curves, therefore we conclude for the A-model all the nonzero contribution to the expectation value for a BRST invariant operator $\mathcal{O}$ comes from the holomorphic curves.

Now consider the B-model and its transformation laws. By setting the variations to be zero, we can see that the only nonzero contribution to the expectation value, comes from the constant maps. This is a result of requiring $\delta\rho^i = -\alpha d\phi^i = 0$, which gives

$$d\phi^i = 0,$$

i.e. the constant maps.

The above reasoning is the fundamental reason why the mirror map is inevitably tied with the Gromov-Witten invariants.

2.5. **Deformation of the action.** We can construct additional observables from the ones we already constructed. By considering more general observables, it has been suggested that the mirror map might have a natural interpretation, and mirror symmetry in higher dimensions might depend on this construction [Wit92]. We start from the observables $\mathcal{O}_{V_a}$, which is a operator-valued zero-form; to emphasize this fact we will denote it as $\mathcal{O}^{(0)}_{V_a}$.

The topological invariance of the theory tells us that the correlation functions of $\mathcal{O}^{(0)}_{V_a}(P)$ are independent of the point P, which means that $\mathcal{O}^{(0)}_{V_a}$ must be closed in the BRST cohomology.

$$d\mathcal{O}^{(0)}_{V_a} = \{Q, \mathcal{O}^{(1)}_{V_a}\},$$

for some operator-valued one-form $\mathcal{O}^{(1)}_{V_a}$. We can also read this equation from right to left, and conclude that $\mathcal{O}^{(1)}_{V_a}$ is a new observable. Similarly, we repeat this procedure and obtain,

$$d\mathcal{O}^{(1)}_{V_a} = \{Q, \mathcal{O}^{(2)}_{V_a}\},$$

where we obtained a new observable $\mathcal{O}^{(2)}_{V_a}$. If we consider the full set of observable for the closed string field theory, we naturally deform the action by,

$$S \mapsto S + \sum_a t_a \int_\Sigma \mathcal{O}^{(2)}_{V_a}.$$

If we started with a operator $\mathcal{O}^{(0)}_{V_a}$ with ghost number q_a, the new observable $\mathcal{O}^{(2)}_{V_a}$ has ghost number $q_a - 2$. Since there is no constraints on q_a, the new action does not necessarily conserve ghost numbers. We can invoke mirror symmetry on this new action, and study mirror symmetry on the generalized action. This has not been fully analyzed yet, but it is easy to see what happens in the simplest case.

The simplest case is when we consider the A-model with $q_a = 2$, which means that the deformed action still conserves ghost numbers. The A-model action before the additional observables reads,

$$S = \int_\Sigma i\{Q, V\} - 2\pi i \int_\Sigma \phi^*(B + iJ),$$

where

$$V = 2\pi g_{ij}(\psi^{\bar{j}}_z \bar{\partial}\phi^i + \partial\phi^{\bar{j}}\psi^i_{\bar{z}}),$$

and $B + iJ \in H^2(Y, \mathbb{C})$ is the complexified Kähler form. Therefore, adding an additional term to the action will correspond to deforming the pull-back of the complexified Kähler form.

3. D-Branes

In this section, we will introduce open strings and D-branes to the theory. The suitable mathematical tool to describe D-branes turns out to be category theory. We will try to cover category theory from the basics, but due to the limited amount of space and the ignorance of the author most of the details will be used without full proof. This section will follow [Asp04] on the physics side, and we will quote the results of [Wei94], [ML98] and [Har77] on the mathematics side.

Category theory is a abstract way to deal with mathematical objects and structures. To eliminate the dependence on the subject we are studying, category theory rephrases most of the mathematical structure in terms of mappings rather than elements or objects. We first state the definition of a category following [Asp04] and [Wei94].

Definition 1 (Category)**.** A *category* $\mathcal{C}$ consists of the following: a class $\mathrm{obj}(\mathcal{C})$ of objects, a set $\mathrm{Hom}_\mathcal{C}(A, B)$ of *morphisms* for every ordered pair (A, B) of objects, an *identity morphism* $\mathrm{id}_A \in \mathrm{Hom}_\mathcal{C}(A, A)$ for every object A, and a composition function

$$\mathrm{Hom}_\mathcal{C}(A, B) \times \mathrm{Hom}_\mathcal{C}(B, C) \to \mathrm{Hom}_\mathcal{C}(A, C),$$

for every ordered triple (A, B, C) of objects. If $f \in \mathrm{Hom}_\mathcal{C}(A, B)$, and $g \in \mathrm{Hom}_\mathcal{C}(B, C)$, the composition is denoted gf. The above data is subject to two axioms:

(1) Associativity axiom: $(hg)f = h(gf)$ for $f \in \mathrm{Hom}_\mathcal{C}(A, B)$, $g \in \mathrm{Hom}_\mathcal{C}(B, C)$ and $h \in \mathrm{Hom}_\mathcal{C}(C, D)$.
(2) Unit axiom: $\mathrm{id}_B f = f = \mathrm{id}_A f$ for $f \in \mathrm{Hom}_\mathcal{C}(A, B)$.

Note that in the definition, it is not specified which objects we are dealing with. The objects can be sets, groups, rings, topological spaces, and for our case D-branes. To see why we need category theory, we will investigate into the A-model, and the B-model. It will be apparent that the A-model is harder to analyze than the B-model, we will mostly state the results of the A-model, while we will try to outline the derivation of the category of the B-model.

3.1. **A-model.** For the A-model with target manifold Y, the D-branes, which we will call A-branes, form a Fukaya category of Y. The details of the A-branes are quite technical and we refer to [Fuk01, FS97, Fuk97, FOOO07, KS01] for the gruesome details[1].

First we argue that the A-branes are classically Lagrangian submanifolds with a flat $U(1)$ bundle. This result is derived by considering the boundary terms of the variation, and the effects of the B-field on the action.

When Σ has a boundary, all of the $N = (2,2)$ supersymmetry cannot be preserved. This is because the boundary conditions force us to have a reflection condition that mixes the left and right moving fermions. This reflection condition with the almost complex structure J requires the dimension of the A-brane to be real dimension 3. Also when one restricts the Kähler form on the A-brane, it vanishes; this is because the almost complex structure J rotates any vector on the A-brane to a vector that is orthogonal to it, and by the definition of Kähler form $\frac{1}{2}g_{LM}J^M_N d\phi^M d\phi^N$. Such submanifold with middle dimension and vanishing symplectic form is called a *Lagrangian submanifold*.

In addition, the B-field forces the A-model to have a flat bundle. When Σ had no boundary, the correlation functions only depend on the cohomology class of B. This is no longer true when Σ has a boundary, the extra freedom can be written in a 1-form A on Y with the following additional term on the action,

$$S_{\partial\Sigma} = -2\pi i \int_{\partial\Sigma} \phi^*(A).$$

The BRST condition constrains the 1-form A, and when $B = 0$ one can show that $F = dA = 0$ [MPR04, HIV00, OOY96, Wit95]. Therefore, classically a A-brane is a Lagrangian submanifold L with flat $U(1)$ bundle.

Now we consider quantum effects. This subject is rather technical and we will only state the results; the details can be found in [Oh93a, Oh93b]. Quantum effects produce anomalies, and the anomaly is related to the Maslov class of the Lagrangian submanifold L. The Maslov class of L should be zero, in order to eliminate the anomalies. Quantum effects also cause destabilization by open string tadpoles. If an A-brane background provides a stable vacuum, then the one-point function $\langle W_a \rangle$ must be zero. Otherwise, the nonzero expectation value would move the D-brane to another location. Therefore, a A-brane vacuum should have zero expectation value for a one-point function. We need to know how to calculate the one-point correlation function to know exactly what the necessary conditions are for a vacuum A-brane. Fortunately, Witten has discovered how to compute $\langle W_a \rangle$ exactly. Restricting to a case of a line bundle, the

[1]Professor Dominic Joyce used to emphasize how gruesome the details are in his lectures.

condition of $\langle W_a \rangle = 0$ is [Wit95],

$$\sum_{\alpha \in I} \exp\left(2\pi i \int_{D_\alpha} (B + iJ) + 2\pi i \oint_{\partial D_\alpha} A\right) [\partial D_\alpha] = 0 \quad \text{in } H_1(L), \tag{4}$$

where the summation is over holomorphic disks D_α with $\partial D_\alpha \subset L$, and $[\partial D_\alpha]$ is the homology class of ∂D_α.

Combining the results and also taking into account that Lagrangian submanifolds are defined up to Hamiltonian deformations [Joy07], we finally obtain the definition of a A-brane.

Definition 2 (A-brane)**.** An *A-brane* is an element of the equivalence class of Lagrangian 3-manifolds in Y modulo Hamiltonian deformations, which satisfies the tadpole cancellation property (4) and has trivial Maslov class.

Now, we know how the objects of the A-brane category look like, we turn our interests onto the morphisms. We would like a category where the A-branes are the objects and the open strings of the A-models are the morphism. To define the morphisms of two A-branes L_a and L_b, this means that we need to look into the Hilbert space of the open strings beginning on L_a and ending on L_b. This Hilbert has a grading related to the ghost number of the open string. Surprisingly, this grading, i.e. ghost number, will play a crucial role in constructing the categories of both A-branes and B-branes. We follow the notation of [Asp04], and denote this Hilbert space as,

$$\text{Hom}^*(L_a, L_b) = \bigoplus_{m \in \mathbb{Z}} \text{Hom}^m(L_a, L_b).$$

It can be shown that the open string states beginning on L_a and ending on L_b, produce a correction to the BRST operator:

$$\{Q, W_{p_q}\} = \sum_b n_{ab} W_{p_a},$$

where n_{ab} is to be determined [Asp04, HKK$^+$03]. Based on this new BRST variation, we can deduce that true Hilbert space will be determined as the Q-cohomology of a complex given by,

$$\cdots \xrightarrow{Q} V_{-1} \xrightarrow{Q} V_0 \xrightarrow{Q} V_1 \xrightarrow{Q} \cdots .$$

We therefore define $\text{Hom}^i(L_a, L_b)$ as the cohomology of this complex at position i.

Since we know what the objects, are and defined the morphisms, we have finally defined a category of A-branes. The category is named after Fukaya who first introduced it. We must emphasized that most of the details were omitted and swept under the rug. The vast amount of technical details were worked out in the last two decades, and can be found in [FS97, Fuk93, Fuk01, KS01, FOOO07]. We state the definition of the category of A-branes,

Definition 3. The *category of A-branes* is the Fukaya category of Y, denoted as $\mathcal{F}(Y)$.

Before we conclude this section, we want to point out one important aspect of A-branes that will also play a key role when discussing B-branes. There is an ambiguity when assigning ghost numbers on A-branes. This ambiguity only let's us define the difference between the ghost numbers of two A-branes. Therefore, the best thing one can do when assigning ghost numbers to an element of $\mathrm{Hom}^i(L_a, L_b)$, which is an open string connecting from L_a to L_b, is to define it as,

$$i + \mu(L_b) - \mu(L_a). \tag{5}$$

As we will see later, this ambiguity will play a crucial role when we discuss the ghost number of B-branes, and ultimately provide enough number of B-branes to invoke mirror symmetry.

3.2. **B-model.** B-branes are simpler than A-branes. Most of the calculations on B-branes reduce to classical expressions, due to the absence of instantons. For this simplicity we will scrutinize B-branes on target space X in more details. We first use a naive approach to B-branes and run into trouble. This will motivate the discussion of categories and sheaves. We will finally use (5) and show that the B-branes form a derived category of coherent sheaves.

3.2.1. *A naive approach.* Using a similar reflection argument as in the A-model, we conclude that the B-branes are holomorphically embedded submanifolds of X. This means that the dimension of the B-branes should be even, i.e. 0, 2, 4, or 6. Initially, we will only consider 6-branes.

We can investigate the effects of the B-field on the B-branes. Setting the B-field to zero and requiring the boundary terms to vanish gives a constraint that says a B-brane $E \to X$ is a holomorphic bundle. Also, when considering only 6-branes one can show that an open string vertex operator for a string stretching from E_1 to E_2 is given by the cohomology groups,

$$H^{0,q}_{\bar{\partial}}(X, \mathrm{Hom}(E_1, E_2)),$$

this is closely tied to the fact that the observables are given by the Dolbeault cohomology, and the Chan-Paton degrees of freedom are associated with $\mathrm{Hom}(E_1, E_2)$ [Asp04]. In contrast to the A-brane case, we choose the ghost number of an operator in this group to be q. In the end this will turn out not to be a good choice and we will restore the ghost number ambiguity similar to the A-model.

It would be wonderful if we could invoke the mirror symmetry and say that the category of A-branes on Y is equivalent to the category of B-branes on X. Unfortunately, we run into immediate trouble. The problem is that we do not have enough of B-branes to match the number of A-branes. Clearly, our assumption that the B-branes are 6-branes is too strong. We will start constructing more B-branes by deforming the known B-branes, by looking at the open strings that connect between different branes. Using this method will require a study of Hilbert spaces of open strings and the language of categories and sheaves.

3.2.2. *Categories and Sheaves.* The objective of this section is to replace the B-branes and open strings, to more algebraic objects namely sheaves and elements of the hyperext group. We start from a simple but powerful observation.

Proposition 4. *There is a one-to-one correspondence between holomorphic vector bundles of rank n on X and locally-free sheaves of rank n on X.*

Proof. The idea of the proof is that when we take a local trivialization, we can naturally associate the holomorphic vector bundle to a locally-free sheaf.

Consider a holomorphic vector bundle over X of rank n, and the structure sheaf ${\mathcal{O}_X}^{\oplus n}$. We may regard $\mathcal{O}_X(U)^{\oplus n}$ as the group of holomorphic sections of the vector bundle over U. This means if we have a vector bundle we can identify it locally over each open set U with a element of $\mathcal{O}_X(U)^{\oplus n}$.

Conversely, if we have a locally-free sheaf $\mathcal{E}$, we have a local isomorphism ϕ_α,

$$\phi_\alpha : \mathcal{E}(U_\alpha) \to \mathcal{O}_X(U_\alpha)^{\oplus n}.$$

On $U_\alpha \cap U_\beta$ by looking at the local isomorphisms, we can obtain the transitions for the holomorphic vector bundle. That is the following map

$$\phi_\beta \phi_\alpha^{-1} : \mathcal{O}_X(U_\alpha \cap U_\beta)^{\oplus n} \to \mathcal{O}_X(U_\alpha \cap U_\beta)^{\oplus n},$$

can be taken as the definition of the transition functions for the holomorphic vector bundle, and therefore a locally-free sheaf gives a holomorphic vector bundle. □

Proposition 4 gives a one-to-one correspondence between a holomorphic vector bundle and a locally-free sheaves. Thus rather than dealing with a holomorphic vector bundle, we will replace it to a locally-free sheaf, which is an algebraic way of describing a holomorphic vector bundle. From now on we will think of a B-brane as a locally-free sheaf rather than a holomorphic vector bundle. Since we have changed a B-brane to a more algebraic object, now it is time to look for candidates for open strings.

In the last subsection, we have seen that an open string beginning on a B-brane E and ending on F, corresponds to an element of,

$$H^{0,q}_{\bar{\partial}}(X, \mathrm{Hom}(E, F)).$$

We will argue that this is actually same as $\mathrm{Ext}^q(\mathcal{E}, \mathcal{F})$. Considering the limited amount of space the full length derivation cannot be discussed here. We will outline the basic steps to obtain such result:

(1) $H^{p,q}_{\bar{\partial}}(X, V) = \check{H}^q(X, \Omega^p \otimes \mathcal{V})$,
where V is a holomorphic vector bundle over X, $\mathcal{V}$ is the locally-free sheaf associated to V, and $\check{H}$ denotes Čech cohomology. This is called Dolbeault's theorem. One can prove this by using the standard spectral sequences [BT82].

(2) $\check{H}^q(X, \Omega^p \otimes \mathcal{V}) = H^q(X, \Omega^p \otimes \mathcal{V})$,
where H is the sheaf cohomology constructed by Grothendieck. This can be proved by using the flabby properties of the injective resolution of the $\mathcal{O}_X$-module $\mathcal{V}$ [Asp04, Har77].

(3) We consider the B-brane case, i.e., when $p = 0$ and $V = \mathrm{Hom}(E, F)$. If we denote the locally-free sheaf that is associated to $\mathrm{Hom}(E, F)$ as $\mathcal{H}om(\mathcal{E}, \mathcal{F})$, by the above two arguments we obtain: $H^{0,q}_{\bar{\partial}}(X, \mathrm{Hom}(E, F)) = H^q(X, \mathcal{H}om(\mathcal{E}, \mathcal{F}))$.

(4) Finally we define, $\mathrm{Ext}^q(\mathcal{E}, \mathcal{F})$ as $H^q(X, \mathcal{H}om(\mathcal{E}, \mathcal{F}))$[2].

By the above argument, we arrive at a definition of a open string.

[2]The definition of Ext is normally stated in terms of right derived functors. Namely, $\mathrm{Ext}^n(\mathcal{E}, \mathcal{F}) := \mathbf{R}^n\mathrm{Hom}(\mathcal{E}, -)(\mathcal{F})$. However, we will be sloppy and will not pursue this definition. [Asp04] gives a very readable explanation about this point, for more technical details consult [Har77].

Definition 5 (Open string)**.** An *open string* from the B-brane associated to the locally-free sheaf $\mathcal{E}$ to another B-brane given by the locally-free sheaf $\mathcal{F}$ is given by an element of the group $\mathrm{Ext}^q(\mathcal{E}, \mathcal{F})$.

3.2.3. *Deformations of B-branes.* We have argued that a B-brane can be replaced with a locally-free sheaf, and a open string can be regarded as a element of the group $\mathrm{Ext}^q(\mathcal{E}, \mathcal{F})$. We finally investigate and try to discover more B-branes to invoke mirror symmetry on the categories. We start from the B-branes we already know and deform it by looking into the candidates where the open string can end.

Since the open string vertex operators integrate over the boundary $\partial\Sigma$, this means the vertex operator should have ghost number 1. In section 3.2.1, we defined the ghost number to be exactly q, therefore we are interested in the elements of $\mathrm{Ext}^1(\mathcal{E}, \mathcal{F})$. However, it turns out that the elements of this group does not generate enough B-branes to invoke mirror symmetry. We need to be a bit more general. We remind ourselves that in the A-brane case there was not enough information to pin down the ghost numbers of the A-branes, the B-branes do not have any justification to have an exact ghost number. Inspired by this observation, we introduce ghost number ambiguity to B-branes and associate the following ghost number, to the group $\mathrm{Ext}^q(\mathcal{E}, \mathcal{F})$,

$$q + \mu(\mathcal{F}) - \mu(\mathcal{E}),$$

which is similar to the A-brane case (5).

We may construct a general collection of B-branes $\mathcal{E}$ in terms of B-branes that are locally-free sheaf $\mathcal{E}^n$ with ghost number n by,

$$\mathcal{E} = \bigoplus_{n\in\mathbb{Z}} \mathcal{E}^n,$$

where n denotes the ghost numbers of the B-brane. By the new ambiguity condition, any element of $\mathrm{Ext}^k(\mathcal{E}^n, \mathcal{E}^{n-k+1})$ has ghost number 1, and thus is included in the deformation. We will study the case when $k = 0$, and obtain the deformation for other cases automatically. When $k = 0$, we are looking at the open string morphism $d_n : \mathcal{E}^n \to \mathcal{E}^{n+1}$ of locally-free sheaves. Explicitly,

$$d = \sum_n d_n$$
$$d_n \in \mathrm{Ext}^0(\mathcal{E}^n, \mathcal{E}^{n+1}) = \mathrm{Hom}(\mathcal{E}^n, \mathcal{E}^{n+1})$$

Let $W_a^{(1)}$ be the operator satisfying,

$$\{Q, W_d^{(1)}\} = d_\Sigma d,$$

where d_Σ is the world sheet exterior differential operator. We deform the action by,

$$S \mapsto S + \oint_{\partial\Sigma} W_d^{(1)}.$$

By the standard Noether procedure, we can construct a conserved charge that corresponds to the deformation of the BRST charge,

$$Q = Q_0 + d.$$

If we impose the nilpotent condition $Q^2 = 0$, we must have,

$$\{Q_0, d\} + d^2 = 0.$$

We assume the deformation d was a open string vertex in our original theory, hence $\{Q_0, d\} = 0$. This assumption can be justified when we are deforming the theory only infinitesimally. This leads to $d^2 = 0$, which makes us inevitably think about complexes. Physically multiplication of d by itself means that we use the operator product algebra for open strings. Using the boundary condition for open strings, and the fact that the operator product is simply the wedge product in the B-model, $d^2 = 0$ implies [Asp04],

$$d_{n+1}d_n = 0 \quad \text{for any } n.$$

In other words, if we construct a collection $\mathcal{E}$ of B-branes $\mathcal{E}^n$, actually $\mathcal{E}$ is a complex,

$$\cdots \xrightarrow{d_{n-1}} \mathcal{E}^n \xrightarrow{d_n} \mathcal{E}^{n+1} \xrightarrow{d_{n+1}} \mathcal{E}^{n+2} \xrightarrow{d_{n+2}} \cdots .$$

We will denote such complexes by $\mathcal{E}^\bullet$ to emphasize it is an complex.

So we have constructed a new type of B-branes out of the known ones. The objects in the category of B-branes appears to be complexes of locally-free sheaves. We will soon see that we need to quotient out a large set of equivalences and eventually introduce derived categories.

3.2.4. *Hilbert space of open strings.* We have deformed our Lagrangian and obtained new B-branes. The deformation also changes the Hilbert space of open strings, and since the open strings give the morphisms, we should investigate in their behavior. As usual, Hilbert spaces are given by the BRST cohomology, and we need to use spectral sequence techniques to calculate the cohomology. We refer to [BT82, Har77] for the mathematical details. We will just outline the procedure.

The simplest case is when a open string begins on a B-brane which is a complex $\mathcal{E}^\bullet$ and ends on a locally-free sheaf $\mathcal{F}$. Suppose $\mathcal{F}$ has an injective resolution,

$$0 \to \mathcal{F} \to \mathcal{I}^0 \to \mathcal{I}^1 \to \mathcal{I}^2 \to \cdots$$

then we can construct a double complex $E_0^{p,q} = \mathrm{Hom}(\mathcal{E}^{-p}, \mathcal{I}^q)$. The spectral sequence construction applied to this complex will give the cohomology of $Q = Q_0 + d$. We will denote this Hilbert space $\mathrm{Ext}^q(\mathcal{E}^\bullet, \mathcal{F})$.

Similarly, we can use the spectral sequence construction on open strings from $\mathcal{E}$ to $\mathcal{F}^\bullet$, and open strings from $\mathcal{E}^\bullet$ to $\mathcal{F}^\bullet$. These cases are more complicated to work out but it can be done [Asp04]. We will denote the Hilbert spaces as $\mathrm{Ext}^n(\mathcal{E}^\bullet, \mathcal{F})$ and $\mathrm{Ext}^n(\mathcal{E}^\bullet, \mathcal{F}^\bullet)$ respectively.

We finally define the morphisms between the B-branes. Since we are considering open strings with ghost number zero, it is sensible to define $\mathrm{Hom}(\mathcal{E}^\bullet, \mathcal{F}^\bullet) = \mathrm{Ext}^0(\mathcal{E}^\bullet, \mathcal{F}^\bullet)$. Now we know how the objects look like and we defined the morphism between the objects, we have defined the category of B-branes!

3.2.5. *The derived category and coherent sheaves.* In principal, we have constructed the category of B-branes. However, to proceed further we would like to analyze it a bit more and extract the mathematical structure of it. We will justify that we should look into the derived category of coherent sheaves denoted as $\mathbf{D}(X)$ in this section.

Given two B-branes $\mathcal{E}^\bullet$ and $\mathcal{F}^\bullet$ how do we know whether they are the same? The answer lies on the Hilbert space of the open strings, namely, $\mathrm{Hom}(\mathcal{E}^\bullet, \mathcal{F}^\bullet) = \mathrm{Ext}^0(\mathcal{E}^\bullet, \mathcal{F}^\bullet)$. Two B-branes $\mathcal{E}^\bullet$ and $\mathcal{F}^\bullet$ are identical when the Hilbert space of open strings between them, i.e. $\mathrm{Hom}(\mathcal{E}^\bullet, \mathcal{F}^\bullet)$, is identical to the Hilbert space of open strings on itself, i.e. $\mathrm{Hom}(\mathcal{F}^\bullet, \mathcal{F}^\bullet)$. This condition is satisfied when two complexes are quasi-isomorphic[3], and we want to quotient out by this relation. This motivates the construction of derived categories.

A derived category $\mathbf{D}(\mathcal{C})$ is constructed from a *abelian category* $\mathcal{C}$, by taking the objects as the complexes of objects of $\mathcal{C}$. The morphisms are given by

(1) chain maps modulo chain homotopies, and
(2) if a morphism f is a quasi-isomorphism, we include f^{-1}.

The second condition naturally includes isomorphisms between quasi-isomorphic objects in the category, and in effect quotient out the quasi-isomorphic objects.

There is still one little glitch in this construction. The category of locally-free sheaves, i.e. the B-branes, are not abelian. This is because locally-free sheaves do not form an abelian category, since they do not contain their own cokernels[4]. We replace the category of locally-free sheaves with an minimal abelian subcategory of $\mathcal{O}_X$-modules containing locally-free sheaves. This is the category of coherent sheaves. We finally define the category of B-branes,

Definition 6. The *category of B-branes* is the derived category of coherent sheaves $\mathbf{D}(X)$.

This was actually first conjectured by Kontsevich [Kon95], and proved by Douglas [Dou01]. A readable detailed treatment can be found in [Asp04].

4. Homological Mirror Symmetry

4.1. **Stability.** It would be nice by this point to invoke mirror symmetry and say that the Fukaya category $\mathcal{F}(Y)$, and the derived category $\mathbf{D}(X)$ are equivalent. However, there is still one more obstacle to sort out. We have constructed too many D-branes after all! This is because actually the D-branes in the topological field theory should correspond to the BPS states of the physical theory before the twist. BPS condition is stronger than what we have discussed for general D-branes, and therefore we need to impose further condition on the D-branes to invoke the full mirror symmetry. This condition is called "stability", and provides contact with the real world. Mathematically, this will lead to triangulated categories.

We will only discuss the case for the A-model. This is because while we could discuss "stability" issues for the B-model, it is similar to the A-model case. In addition, for the A-model, we have a nice physical picture of what is happening to the A-branes.

First we recall definition 2 of an A-brane, which states that A-branes are Lagrangian submanifolds. However, not every Lagrangian submanifolds corresponds to the BPS states. As first observed in [BBS95], it is the special Lagrangians, i.e. calibrated submanifolds, that corresponds to the BPS states.

[3]A quasi-isomorphism is a chain map f which the induced morphisms f_*^n of the cohomology of the complexes are isomorphisms in the category for all n.

[4]Thanks to Vicky for explaining this point.

Even though the mathematical details can be challenging, it is easy to motivate the idea that the special Lagrangians are the BPS states. If we think as the D-brane as a kind of membrane with tension, BPS states would be the stable ones where the volume of the brane is minimal. This means that BPS states should correspond to the volume minimizing Lagrangian submanifold, such submanifolds are called special Lagrangian submanifolds. This is the physical reasoning of correspondence between the special Lagrangian submanifolds and the BPS states. This aspect of the theory is also related to calibrated geometry, a good reference in this field is [Joy07].

If we recall once again the original definition 2 of A-branes, we might guess that every Lagrangian can be deformed to a special Lagrangian by Hamiltonian deformation. However, this is not true. It turns out most of the Lagrangians have no special Lagrangian equivalent to them, and hence go under a "decay" into a stable BPS state. A detailed and explicit mathematical description of such process can be found in [Joy03].

How could we incorporate this in to the language of categories? It can be showed that we need to triangulate the original categories [Asp04].

4.2. **Homological mirror symmetry.** We are finally ready to state the homological mirror symmetry conjecture! We consider triangulated Fukaya category $\mathrm{Tr}\mathcal{F}(Y)$ and triangulated derived category $\mathbf{D}(X)$, the homological mirror symmetry conjecture states,

Conjecture 7 (Homological mirror symmetry)**.** *If X and Y are mirror Calabi-Yau threefolds then the category $\mathbf{D}(X)$ is equivalent to the category $\mathrm{Tr}\mathcal{F}(Y)$.*

This remarkable conjecture has been verified for 2-tori [PZ98], and quartic K3 surfaces [Sei03]. However, it still stands firm to be proved for general Calabi-Yau threefolds.

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