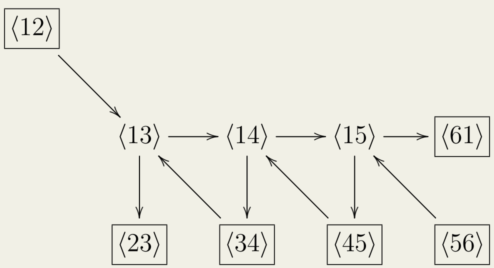
Scattering Amplitudes and Cluster Polylogarithms



Anastasia Volovich
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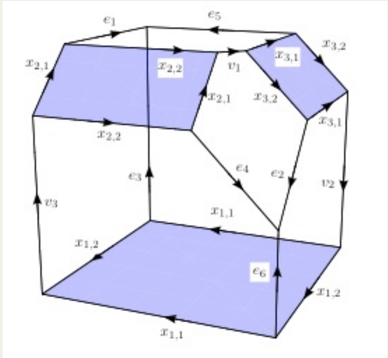


Oxford, September 2014

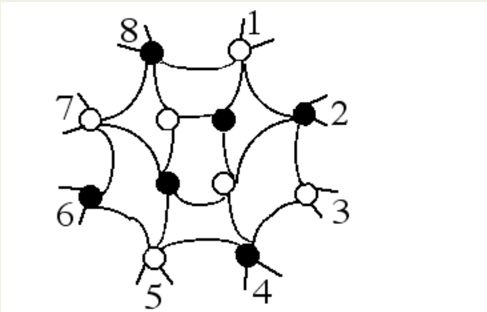
ArXiv: 1305.1617, 1401.6446, 1406.2055

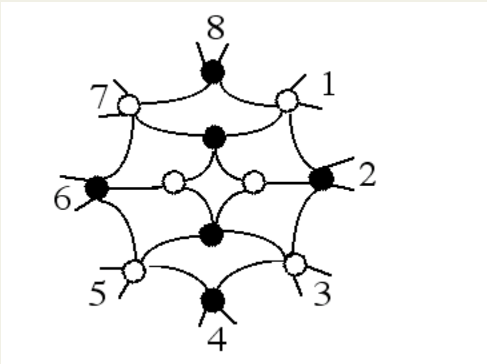
Golden, Goncharov, Paulos, Spradlin, Vergu

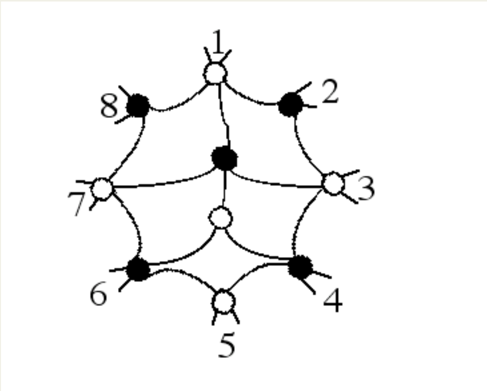


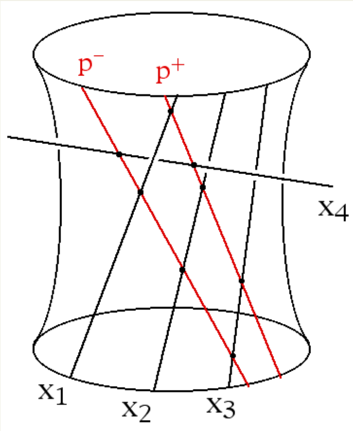


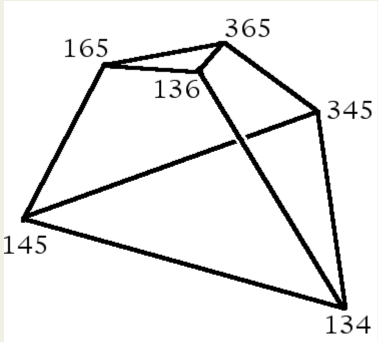
Happy Birthday, Andrew!

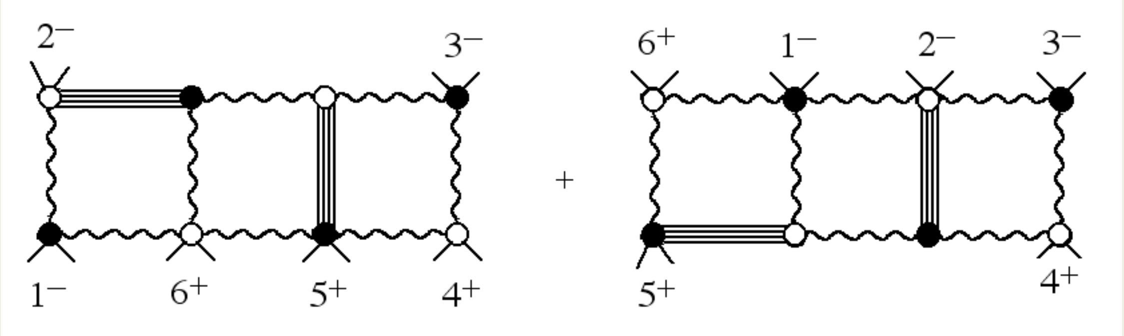












In my talk, I will

- explore cluster algebra structure of amplitudes in N=4 Yang-Mills (which we observed experimentally)
- explain how to use it to compute 2-loop all-n MHV amplitudes

Plan

- Introduction
- Coproduct and amplitudes: functions and arguments
- Cluster algebras basics
- Cluster polylogs as amplitudes building blocks
- Conclusion

2-loop 6-point MHV amplitude in N=4 SYM

$$R_6^{(2)} = \sum_{\text{cyclic}} \text{Li}_4\left(-\frac{\langle 1234\rangle\langle 2356\rangle}{\langle 1236\rangle\langle 2345\rangle}\right) - \frac{1}{4}\text{Li}_4\left(-\frac{\langle 1246\rangle\langle 1345\rangle}{\langle 1234\rangle\langle 1456\rangle}\right)$$

$+$ products of $\text{Li}_k(-x)$ functions of lower weight

Goncharov, Spradlin, Vergu, AV

1. Functions: only classical polylogs degree 4 appear

$$Li_k(z) = \int_0^z Li_{k-1}(t)\, d\log t \qquad Li_1(z) = -\log(1-z)$$

2-loop 6-point MHV Amplitude

$$R_6^{(2)} = \sum_{\text{cyclic}} \text{Li}_4\left(-\frac{\langle 1234\rangle\langle 2356\rangle}{\langle 1236\rangle\langle 2345\rangle}\right) - \frac{1}{4}\text{Li}_4\left(-\frac{\langle 1246\rangle\langle 1345\rangle}{\langle 1234\rangle\langle 1456\rangle}\right)$$

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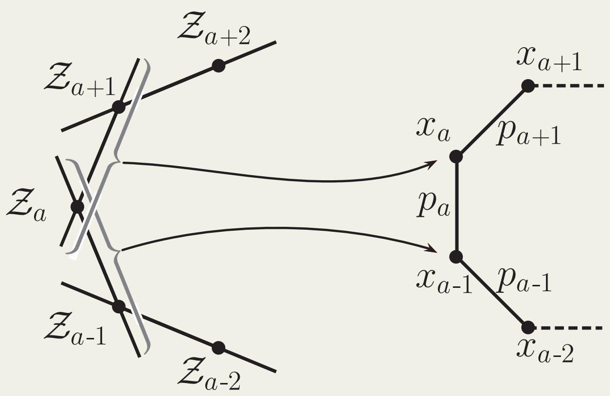
1. Functions: only classical polylogs appear

$$Li_k(z) = \int_0^z Li_{k-1}(t)\, d\log t \qquad Li_1(z) = -\log(1-z)$$

2. Arguments:

Kinematics

- Kinematics of an n-point amplitude can be described in terms of n momentum twistors



$$Z = (\lambda_\alpha, x_{\alpha\dot{\alpha}}\lambda^\alpha)$$

Hodges

Null momentum

$$p_a^\mu \mapsto (p_a)_{\underline{\alpha}\,\underline{\dot{\alpha}}} \equiv p_a^\mu(\sigma_\mu)_{\underline{\alpha\dot{\alpha}}} \equiv \lambda_{\underline{\alpha}}^{(a)}\widetilde{\lambda}_{\underline{\dot{\alpha}}}^{(a)}$$

Momentum conservation

$$p_a \equiv x_a - x_{a-1}$$

- Dual conformal objects are ratios of 4-brackets

$$\langle ijkl\rangle := \det(Z_i Z_j Z_k Z_l),$$

Drummond, Henn, Korchemsky, Sokachev

- Amplitudes are functions on 3(n-5) dim space

$$Conf_n(CP^3) = Gr(4,n)/(C^*)^n$$

2-loop 6-point MHV Amplitude

$$R_6^{(2)} = \sum_{\text{cyclic}} \text{Li}_4\left(-\frac{\langle 1234\rangle\langle 2356\rangle}{\langle 1236\rangle\langle 2345\rangle}\right) - \frac{1}{4}\text{Li}_4\left(-\frac{\langle 1246\rangle\langle 1345\rangle}{\langle 1234\rangle\langle 1456\rangle}\right)$$

$+$ products of $\text{Li}_k(-x)$ functions of lower weight

Goncharov, Spradlin, Vergu, AV

1. Functions: only classical polylogs appear

$$Li_k(z) = \int_0^z Li_{k-1}(t)\, d\log t \qquad Li_1(z) = -\log(1-z)$$

2. Arguments: 9 out of 45 cross-ratios appear

$$v_1 = \frac{\langle 1246\rangle\langle 1345\rangle}{\langle 1234\rangle\langle 1456\rangle}, \quad v_2 = \frac{\langle 1235\rangle\langle 2456\rangle}{\langle 1256\rangle\langle 2345\rangle}, \quad v_3 = \frac{\langle 1356\rangle\langle 2346\rangle}{\langle 1236\rangle\langle 3456\rangle},$$

$$x_1^+ = \frac{\langle 1456\rangle\langle 2356\rangle}{\langle 1256\rangle\langle 3456\rangle}, \quad x_2^+ = \frac{\langle 1346\rangle\langle 2345\rangle}{\langle 1234\rangle\langle 3456\rangle}, \quad x_3^+ = \frac{\langle 1236\rangle\langle 1245\rangle}{\langle 1234\rangle\langle 1256\rangle},$$

$$x_1^- = \frac{\langle 1234\rangle\langle 2356\rangle}{\langle 1236\rangle\langle 2345\rangle}, \quad x_2^- = \frac{\langle 1256\rangle\langle 1346\rangle}{\langle 1236\rangle\langle 1456\rangle}, \quad x_3^- = \frac{\langle 1245\rangle\langle 3456\rangle}{\langle 1456\rangle\langle 2345\rangle},$$

$$u_a = \frac{(p_a + p_{a+1})^2 (p_{a+3} + p_{a+4})^2}{(p_a + p_{a+1} + p_{a+2})^2 (p_{a+2} + p_{a+3} + p_{a+4})^2}$$

$$v_a = \frac{1}{u_a} - 1$$

$$x_a^\pm = \frac{u_a}{2u_1u_2u_3}\left(u_1 + u_2 + u_3 - 1 \pm \sqrt{(u_1 + u_2 + u_3 - 1)^2 - 4u_1u_2u_3}\right)$$

Natural questions

- Why does the remainder function contain only classical polylogs ?
- Why do these particular arguments appear ?
- How to generalize this formula?

$$\langle a(bc)(de)(fg)\rangle \equiv \langle abde\rangle\langle acfg\rangle - \langle abfg\rangle\langle acde\rangle,$$
$$\langle ab(cde)\cap(fgh)\rangle \equiv \langle acde\rangle\langle bfgh\rangle - \langle bcde\rangle\langle afgh\rangle,$$

I will focus in my talk on 2-loops all n.

For higher loops and NMHV: see very impressive work by Dixon, Drummond, Duhr, Pennington, von Hippel

Coproduct δ

Goncharov

To every transcendental function degree 4 associate element of

$$B_2 \wedge B_2 \quad \text{and} \quad B_3 \otimes C^*$$

coproduct δ

This is what characterizes the degree 4 function modulo products of lower weight functions.

The first determines function and

the second arguments.

Symbol

The symbol == an element of the k-fold tensor product of the multiplicative group of rational functions defined recursively

$$T_k \to S(T_k) = R_1 \otimes \cdots \otimes R_k$$

$$dT_k = \sum_i T^i_{k-1} d\log R_i \to S(T_k) = \sum_i S(T^i_{k-1}) \otimes R_i$$

$$\log R \to R; \;\; \log R_1 \log R_2 \to R_1 \otimes R_2 + R_2 \otimes R_1; \;\; \mathrm{Li}_2(R) \to -(1-R) \otimes R$$

Symbol trivializes polylog identities

$$Li_2(x) + Li_2(-x) = \tfrac{1}{2} Li_2(x^2)$$

$$-(1-x) \otimes x - (1+x) \otimes (-x) = -(1-x^2) \otimes x = -\tfrac{1}{2}(1-x^2) \otimes x^2$$

Symbol $\Rightarrow$ $B_2 \wedge B_2$

- Antisymmetrize symbol

$$a \otimes b \otimes c \otimes d \to (a \wedge b) \wedge (c \wedge d)$$

- Theorem. Function is a classical polylog iff this is zero. [Goncharov]

This object is an element of Bloch group $B_2 \wedge B_2$

Symbol ➡ $B_3 \otimes C^*$

- Recall operators ρ_n which annihilate products of lower-weight functions.

$$\rho_1 = \mathrm{id},$$
$$\rho_n(a_1 \otimes \cdots \otimes a_n) = \rho_{n-1}(a_1 \otimes \cdots \otimes a_{n-1}) \otimes a_n - \rho_{n-1}(a_2 \otimes \cdots \otimes a_n) \otimes a_1.$$

- Given a weight 4 polylog, use this map as follows

$$\delta(a_1 \otimes a_2 \otimes a_3 \otimes a_4)|_{\Lambda^2 \mathrm{B}_2} = \rho(a_1 \otimes a_2) \bigwedge \rho(a_3 \otimes a_4),$$
$$\delta(a_1 \otimes a_2 \otimes a_3 \otimes a_4)|_{\mathrm{B}_3 \otimes \mathbb{C}^*} = \rho(a_1 \otimes a_2 \otimes a_3) \bigotimes a_4 - \rho(a_2 \otimes a_3 \otimes a_4) \bigotimes a_1$$

Examples

$$\delta\,\mathrm{Li}_4(x)|_{\Lambda^2\,\mathrm{B}_2} = 0,$$
$$\delta\,\mathrm{Li}_4(x)|_{\mathrm{B}_3\,\otimes\mathbb{C}^*} = -\{-x\}_3 \otimes x$$

$$\delta\{x\}_k = \begin{cases} (1+x)\bigwedge x & k=2, \\ \{x\}_{k-1}\bigotimes x & k>2. \end{cases}$$

Elements of B_k are finite linear combinations of $\{x\}_k$

$$\mathrm{Symbol}[Li_{2,2}(x,y) = \sum_{0<n<m} \frac{x^n}{n^2}\frac{y^m}{m^2}] =$$

$$\begin{aligned}
&(y-1)\otimes(x-1)\otimes x\otimes y+(y-1)\otimes(x-1)\otimes y\otimes x+\\
&(y-1)\otimes y\otimes(x-1)\otimes x-(xy-1)\otimes(x-1)\otimes x\otimes y-\\
&(xy-1)\otimes(x-1)\otimes y\otimes x-(xy-1)\otimes x\otimes(x-1)\otimes x+\\
&(xy-1)\otimes x\otimes x\otimes x+(xy-1)\otimes x\otimes x\otimes y+\\
&(xy-1)\otimes x\otimes(y-1)\otimes y+(xy-1)\otimes x\otimes y\otimes x+\\
&(xy-1)\otimes(y-1)\otimes x\otimes y+(xy-1)\otimes(y-1)\otimes y\otimes x-\\
&(xy-1)\otimes y\otimes(x-1)\otimes x+(xy-1)\otimes y\otimes x\otimes x+\\
&(xy-1)\otimes y\otimes(y-1)\otimes y.
\end{aligned}$$

$$\delta\mathrm{Li}_{2,2}(x,y)|_{B_2\wedge B_2} = \{-y\}_2\wedge\{-x\}_2 - \{-xy\}_2\wedge\{-x\}_2 + \{-xy\}_2\wedge\{-y\}_2,$$

$$\begin{aligned}
\delta\mathrm{Li}_{2,2}(x,y)|_{B_3\otimes\mathbb{C}^*} &= \{-x\}_3\otimes y - 2\{-x\}_3\otimes(xy-1) - \{-y\}_3\otimes x + 2\{-y\}_3\otimes(xy-1)\\
&-\left\{\frac{1-x}{xy-1}\right\}_3\otimes x + \left\{\frac{1-y}{xy-1}\right\}_3\otimes y - \{xy-1\}_3\otimes y - \left\{\frac{xy}{1-xy}\right\}_3\otimes x\\
&-\left\{\frac{x(1-y)}{xy-1}\right\}_3\otimes y + \left\{\frac{(1-x)y}{xy-1}\right\}_3\otimes x + \{x-1\}_3\otimes x - \{y-1\}_3\otimes y
\end{aligned}$$

2-loop 6-point MHV

$$B_3 \otimes C^* \quad \sum_{i=1}^{3} \{x_i^+\}_3 \otimes x_i^+ + \{x_i^-\}_3 \otimes x_i^- - \frac{1}{2}\{v_i\}_3 \otimes v_i.$$

$$v_1 = \frac{\langle 1246\rangle\langle 1345\rangle}{\langle 1234\rangle\langle 1456\rangle}, \quad v_2 = \frac{\langle 1235\rangle\langle 2456\rangle}{\langle 1256\rangle\langle 2345\rangle}, \quad v_3 = \frac{\langle 1356\rangle\langle 2346\rangle}{\langle 1236\rangle\langle 3456\rangle},$$

$$x_1^+ = \frac{\langle 1456\rangle\langle 2356\rangle}{\langle 1256\rangle\langle 3456\rangle}, \quad x_2^+ = \frac{\langle 1346\rangle\langle 2345\rangle}{\langle 1234\rangle\langle 3456\rangle}, \quad x_3^+ = \frac{\langle 1236\rangle\langle 1245\rangle}{\langle 1234\rangle\langle 1256\rangle},$$

$$x_1^- = \frac{\langle 1234\rangle\langle 2356\rangle}{\langle 1236\rangle\langle 2345\rangle}, \quad x_2^- = \frac{\langle 1256\rangle\langle 1346\rangle}{\langle 1236\rangle\langle 1456\rangle}, \quad x_3^- = \frac{\langle 1245\rangle\langle 3456\rangle}{\langle 1456\rangle\langle 2345\rangle},$$

These are Fock-Goncharov coordinates for A3 cluster algebra as I will explain momentarily.

$$B_2 \wedge B_2 = 0$$

2-loop n-point MHV Symbol

Caron-Huot

The differential of the n-point function is expressed as

$$dR_n = \sum_{i,j} C_{i,j} d\log\langle i{-}1\, i\, i{+}1\, j\rangle \tag{A.1}$$

where $C_{2,i}$ is the sum of the four contributions

$$\begin{aligned}
C_{2,i}^{(1)} &= \log u_{2,i-1,i,1} \times \sum_{j=2}^{i-1}\sum_{k=i}^{n+1}\left[\mathrm{Li}_2(1-u_{j,k,k-1,j+1}) + \log\frac{x_{j,k}^2}{x_{j+1,k}^2}\log\frac{x_{j,k}^2}{x_{j,k-1}^2}\right],\\
C_{2,i}^{(2)} &= \sum_{j=4}^{i-2}\Delta(1,2;j{-}1,j;i{-}1,i),\\
C_{2,i}^{(3)} &= \sum_{j=i+2}^{n}\Delta(2,1;j,j{-}1;i,i{-}1),\\
C_{2,i}^{(4)} &= -2\mathrm{Li}_3(1-\tfrac{1}{u}) - \mathrm{Li}_2(1-\tfrac{1}{u})\log u - \frac{1}{6}\log^3 u + \frac{\pi^2}{6}\log u,
\end{aligned} \tag{A.2}$$

and other $C_{i,j}$ are obtained by cyclic symmetry. In the first line, $x_{j+1} \equiv x_2$ when $j = i{-}1$, and $x_{k-1} \equiv x_1$ when $k = i$, and in the last line, $u = u_{2,i-1,i,1}$. The symbol of Δ is

$$\begin{aligned}
&\mathcal{S}\Delta(1,2;j{-}1,j;i{-}1,i)\\
&= \left(\mathcal{S}[I_5(i;1,2;j{-}1,j)] \otimes \frac{\langle i\,i{+}1(\bar{2})\cap(\bar{j})\rangle\langle 23ij\rangle}{\langle j{-}1\,j\,j{+}1\,i\rangle\langle 123j\rangle\langle 23\,i\,i{+}1\rangle} - ((i\,i{+}1)\to(i{-}1\,i))\right)\\
&+ \left(\begin{aligned}
&\tfrac{1}{2}\mathcal{S}[\mathrm{Li}_2(1-u_{j,2,1,i-1}) - \mathrm{Li}_2(1-u_{j,2,1,i})] \otimes \left(\frac{\langle 123i\rangle\langle j{-}1\,j\,j{+}1\,2\rangle\langle 23ij\rangle}{\langle 123j\rangle\langle j{-}1\,j\,j{+}1\,i\rangle\langle 23\,i\,i{+}1\rangle}\right)^2 \frac{\langle j\,j{+}1(\bar{2})\cap(\bar{i})\rangle\langle i\,i{+}1\,j\,j{+}1\rangle}{\langle 2\,i\,j\,j{+}1\rangle\langle 13(2\,i{-}1\,i)\cap(2\,j\,j{+}1)\rangle}\\
&+\tfrac{1}{2}\mathcal{S}[\mathrm{Li}_2(1-u_{j,i-1,i,2}) - \mathrm{Li}_2(1-u_{j,i-1,i,1})] \otimes \left(\frac{\langle 12\,i{-}1\,i\rangle\langle 23ij\rangle}{\langle 123i\rangle\langle i{-}1\,i\,i{+}1\,j\rangle\langle 23\,i{-}1\,i\rangle}\right)^2 \frac{\langle j\,j{+}1(\bar{2})\cap(\bar{i})\rangle\langle i{-}1\,i\,i{+}1(i23)\cap(i\,j\,j{+}1)\rangle}{\langle 2\,i\,j\,j{+}1\rangle\langle 12\,j\,j{+}1\rangle}\\
&+\tfrac{1}{2}\mathcal{S}[\mathrm{Li}_2(1-u_{2,i-1,i,1})] \otimes \frac{\langle j\,j{+}1(\bar{2})\cap(\bar{i})\rangle\langle i{-}1\,i\,i{+}1(i23)\cap(i\,j\,j{+}1)\rangle}{\langle 2\,i\,j\,j{+}1\rangle\langle 13(2\,i{-}1\,i)\cap(2\,j\,j{+}1)\rangle}\\
&+\tfrac{1}{2}\mathcal{S}[\log u_{j,i-1,i,2}\log u_{j,2,1,i-1}] \otimes \left(\frac{\langle 23ij\rangle}{\langle 123j\rangle}\right)^2 \frac{\langle j\,j{+}1(\bar{2})\cap(\bar{i})\rangle\langle 13(2\,i{-}1\,i)\cap(2\,j\,j{+}1)\rangle}{\langle 2\,i\,j\,j{+}1\rangle\langle 23\,i{-}1\,i\rangle\langle i{-}1\,i\,i{+}1(i23)\cap(i\,j\,j{+}1)\rangle}\\
&-((j\,j{+}1)\to(j{-}1\,j))
\end{aligned}\right)\\
&+ \mathcal{S}[I_5(1;i{-}1,i;j{-}1,j)] \otimes \frac{\langle 12ij\rangle\langle 23\,i{-}1\,i\rangle}{\langle 12\,i{-}1\,i\rangle\langle 23ij\rangle}\\
&+ \mathcal{S}[\log u_{i,j-1,j,1}\log u_{2,i-1,i,1}] \otimes \frac{\langle j{-}1\,j\,j{+}1(j12)\cap(j\,i\,i{+}1)\rangle\langle 123i\rangle\langle 23\,i{-}1\,i\rangle}{\langle 123j\rangle\langle j{-}1\,j\,j{+}1\,i\rangle\langle 12\,i{-}1\,i\rangle\langle 23\,i\,i{+}1\rangle}.
\end{aligned} \tag{A.3}$$

2-loop 7-point MHV: coproduct

$$\delta(R_7^{(2)})|_{B_2\wedge B_2} = \left\{\frac{\langle 6(17)(23)(45)\rangle}{\langle 1267\rangle\langle 3456\rangle}\right\}_2 \bigwedge \left\{-\frac{\langle 5(17)(23)(46)\rangle}{\langle 1567\rangle\langle 2345\rangle}\right\}_2 + \left\{\frac{\langle 1234\rangle\langle 2357\rangle}{\langle 1237\rangle\langle 2345\rangle}\right\}_2 \bigwedge \left\{-\frac{\langle 5(17)(23)(46)\rangle}{\langle 1567\rangle\langle 2345\rangle}\right\}_2$$
$$+ \left\{\frac{\langle 1256\rangle\langle 4567\rangle}{\langle 1567\rangle\langle 2456\rangle}\right\}_2 \bigwedge \left\{\frac{\langle 1235\rangle\langle 2456\rangle}{\langle 1256\rangle\langle 2345\rangle}\right\}_2 + \text{ dihedral} + \text{parity}$$

$$\delta(R_7^{(2)})|_{B_3\otimes\mathbb{C}^*} = X \otimes \frac{\langle 1234\rangle\langle 3567\rangle}{\langle 1237\rangle\langle 3456\rangle} + \frac{1}{2} Y \otimes \frac{\langle 1567\rangle\langle 2345\rangle\langle 3467\rangle}{\langle 1237\rangle\langle 3456\rangle\langle 4567\rangle} + \text{ dihedral} + \text{parity}$$

(where

$$X = \left\{\frac{\langle 1234\rangle\langle 1267\rangle\langle 1567\rangle\langle 3456\rangle}{\langle 1256\rangle\langle 1346\rangle\langle 7(12)(34)(56)\rangle}\right\}_3 - \left\{\frac{\langle 1234\rangle\langle 1567\rangle\langle 3467\rangle}{\langle 1346\rangle\langle 7(12)(34)(56)\rangle}\right\}_3 + \left\{\frac{\langle 1267\rangle\langle 1347\rangle\langle 3456\rangle}{\langle 1346\rangle\langle 7(12)(34)(56)\rangle}\right\}_3 + \ldots$$
$$Y = \left\{\frac{\langle 1234\rangle\langle 1267\rangle\langle 4567\rangle}{\langle 1247\rangle\langle 6(12)(34)(57)\rangle}\right\}_3 - \left\{\frac{\langle 1236\rangle\langle 4567\rangle}{\langle 6(12)(34)(57)\rangle}\right\}_3 - \left\{\frac{\langle 1234\rangle\langle 1267\rangle\langle 3567\rangle}{\langle 1237\rangle\langle 6(12)(34)(57)\rangle}\right\}_3 + \ldots$$
)

$$\langle a(bc)(de)(fg)\rangle \equiv \langle abde\rangle\langle acfg\rangle - \langle abfg\rangle\langle acde\rangle$$

Is there a math structure? How to integrate this?

Now we would like to establish a connection between

amplitudes and

cluster algebras.

Cluster Algebras

- Cluster algebras were first discovered and developed by Fomin and Zelevinski (2002).
- Very informally: commutative algebras constructed from distinguished generators (cluster variables) grouped into disjoint sets of constant cardinality (clusters) which are constructed recursively from the initial cluster by mutations.
- Cluster algebra portal:

http://www.math.lsa.umich.edu/~fomin/cluster.html

A_2 Cluster Algebra

- cluster variables: a_m for $m \in \mathbb{Z}$, subject to $a_{m-1}a_{m+1} = 1 + a_m$
- rank: 2
- clusters: $\{a_m, a_{m+1}\}$ for $m \in \mathbb{Z}$
- initial cluster: $\{a_1, a_2\}$
- mutation: $\{a_{m-1}, a_m\} \to \{a_m, a_{m+1}\}$.

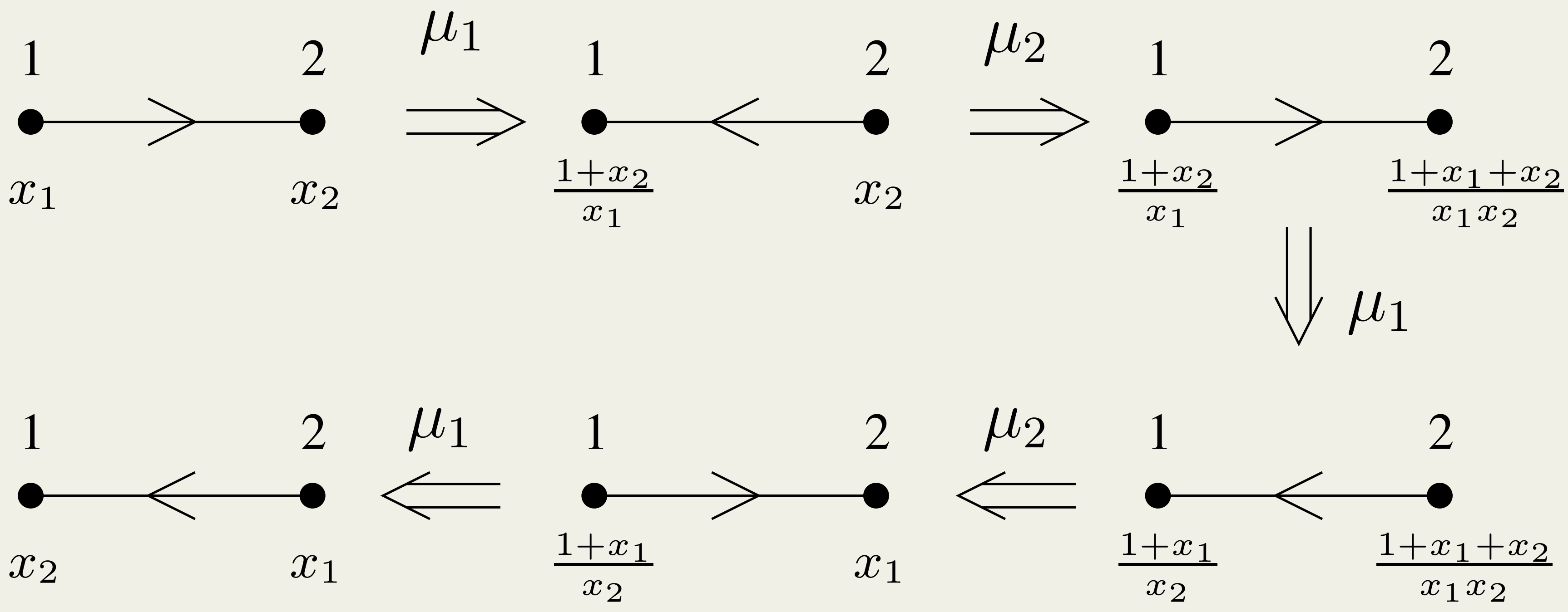
Sequence with period 5 => 5 cluster variables

$$a_1, a_2, a_3 = \frac{1+a_2}{a_1}, a_4 = \frac{1+a_1+a_2}{a_1 a_2}, a_5 = \frac{1+a_1}{a_2}, a_6 = a_1, a_7 = a_1.$$

These are the arguments in Abel identity $\sum_{i=1}^{5} Li_2(-a_i) = 0$

A_2 Quiver

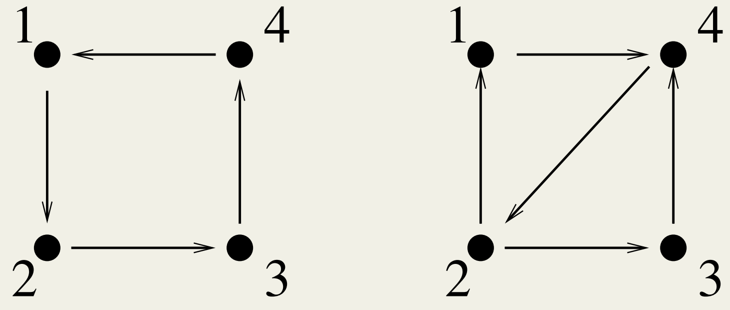
We can represent this using quivers and mutations at each vertex:



Quivers and Mutations

- We can define cluster algebra by a quiver: oriented graph without loops and 2-cycles.
- Given a quiver, get a new one by mutation rule:
  - for each path $i \to k \to j$, add an arrow $i \to j$,
  - reverse all arrows on the edges incident with k,
  - and remove any two-cycles that may have formed.

For vertex 1:



Quivers and Cluster Coordinates

We can encode a quiver by a skew-symmetric matrix

$$b_{ij} = (\#\text{arrows } i \to j) - (\#\text{arrows } j \to i).$$

To each vertex i associate variable a_i

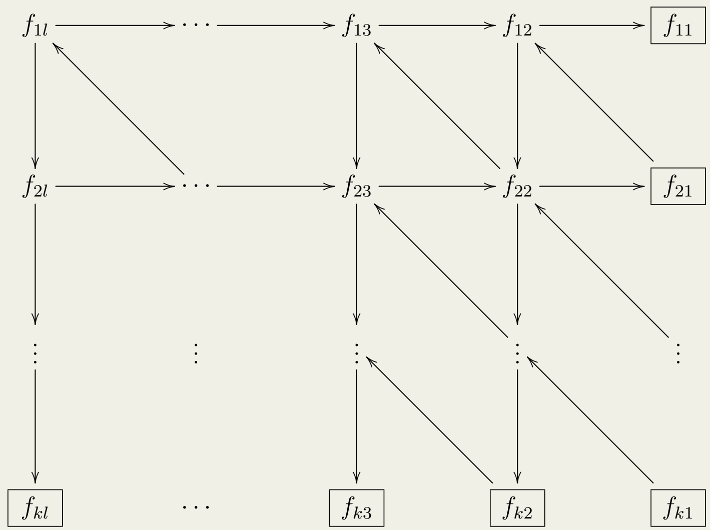
Use matrix b to define mutation relation at vertex k

$$a_k a'_k = \prod_{i|b_{ik}>0} a_i^{b_{ik}} + \prod_{i|b_{ik}<0} a_i^{-b_{ik}},$$

In practice: see Keller Java program

Grassmannian cluster algebras

- Scott (2003) classified all Grassmannian cluster algebras of finite type.



Quiver for

$$Gr(k, k+l)$$

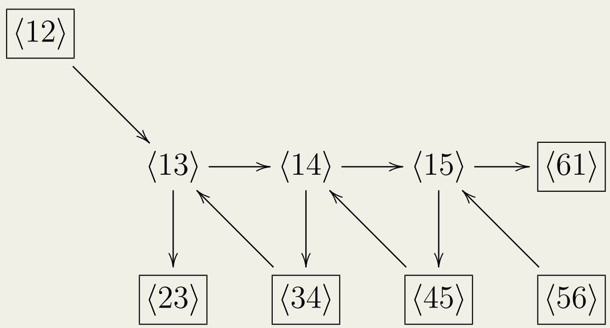
$$f_{ij} = \begin{cases} \frac{\langle i+1,\ldots,k,k+j,\ldots,i+j+k-1\rangle}{\langle 1,\ldots,k\rangle}, & i \le l-j+1, \\ \frac{\langle 1,\ldots,i+j-l-1,i+1,\ldots,k,k+j,\ldots,n\rangle}{\langle 1,\ldots,k\rangle}, & i > l-j+1 \end{cases}.$$

- Amplitudes are functions on $Gr(4,n)/(C^*)^n$
- We have 3 x (n-5) initial quiver (k=4, l=n-4) with initial cluster variables which we then mutate to obtain all cluster coordinates.

Examples: n=6 & n=7

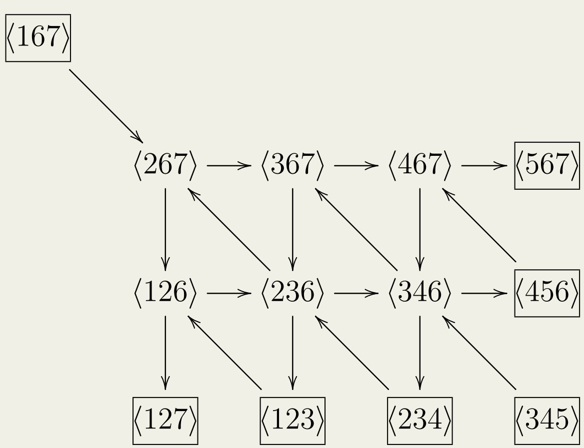
Fomin, Zelevinsky, Scott

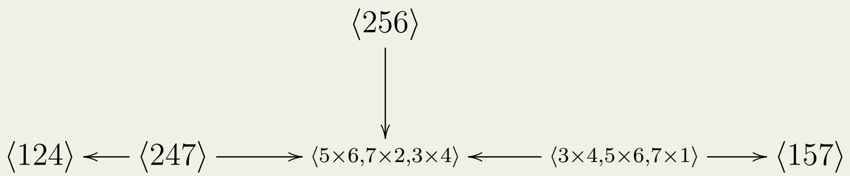
Quivers for cluster algebras of finite type can be turned into Dynkin diagrams by mutations



$$Gr(4,6) = Gr(2,6) \to A_3$$

$$Gr(4,7) = Gr(3,7) \to E_6$$





A and X-coordinates

- In previous examples, initial clusters were labeled by Plucker coordinates $< i_1 \ldots i_k >$
- These are called A-coordinates.
- They are not invariant under rescaling of individual vectors.
- Instead define X-coordinates for each "unfrozen" node: $$X_i = \prod_j a_j^{b_{ij}}.$$ Fock, Goncharov

  cross-ratios
- Mutation at vertex k:

$$X_i' = \begin{cases} X_k^{-1}, & i = k, \\ X_i (1 + X_k^{\operatorname{sgn} b_{ik}})^{b_{ik}}, & i \neq k \end{cases}.$$

2-loop 6-point & A_3 cluster algebra

- Start with quiver. Generate all coordinates by mutations. Mutation generates 14 clusters.

$$\begin{array}{llll}
\langle 13\rangle,\langle 14\rangle,\langle 15\rangle, & \langle 14\rangle,\langle 15\rangle,\langle 24\rangle, & \langle 13\rangle,\langle 15\rangle,\langle 35\rangle, & \langle 13\rangle,\langle 14\rangle,\langle 46\rangle,\\
\langle 15\rangle,\langle 24\rangle,\langle 25\rangle, & \langle 14\rangle,\langle 24\rangle,\langle 46\rangle, & \langle 15\rangle,\langle 25\rangle,\langle 35\rangle, & \langle 13\rangle,\langle 35\rangle,\langle 36\rangle,\\
\langle 13\rangle,\langle 36\rangle,\langle 46\rangle, & \langle 24\rangle,\langle 25\rangle,\langle 26\rangle, & \langle 24\rangle,\langle 26\rangle,\langle 46\rangle, & \langle 25\rangle,\langle 26\rangle,\langle 35\rangle,\\
\langle 26\rangle,\langle 35\rangle,\langle 36\rangle, & \langle 26\rangle,\langle 36\rangle,\langle 46\rangle. & &
\end{array}$$

- 15 A-coordinates: 6 fixed <ii+1>; 9 unfixed <ij>
- 15 X-coordinates:

$$r(i,j,k,l) = -\frac{\langle ij\rangle\langle kl\rangle}{\langle jk\rangle\langle li\rangle}$$

$$\langle ij\rangle = \frac{1}{4!}\epsilon_{ijklmn}\langle klmn\rangle$$

$$\begin{array}{rlrlrl}
v_1 &= r(3,5,6,2), & v_2 &= r(1,3,4,6), & v_3 &= r(5,1,2,4),\\
x_1^+ &= r(2,3,4,1), & x_2^+ &= r(6,1,2,5), & x_3^+ &= r(4,5,6,3),\\
x_1^- &= r(1,4,5,6), & x_2^- &= r(5,2,3,4), & x_3^- &= r(3,6,1,2),\\
e_1 &= r(1,2,3,5), & e_2 &= r(2,3,4,6), & e_3 &= r(3,4,5,1),\\
e_4 &= r(4,5,6,2), & e_5 &= r(5,6,1,3), & e_6 &= r(6,1,2,4),
\end{array}$$

- Note: The top 9/15 are exactly the arguments in 2-loop 6-point amplitude!

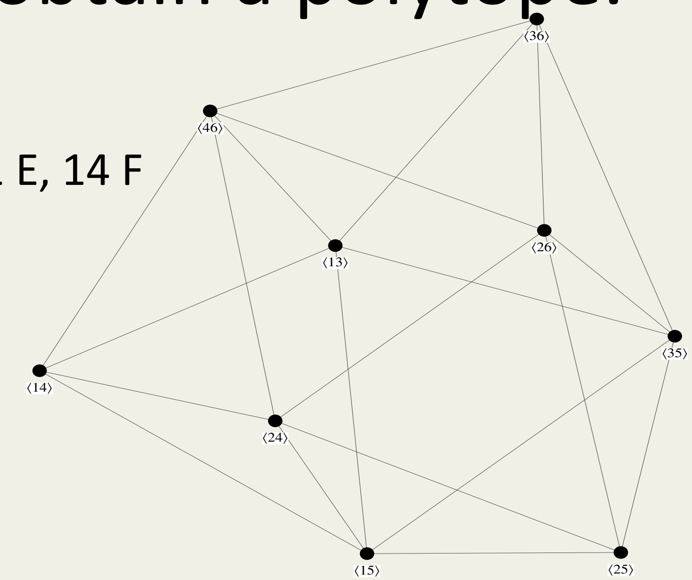
with Golden, Goncharov, Spradlin, Vergu

Geometrically: Stasheff polytopes

- Unfrozen vertices of rank r cluster algebras = vertices of (r-1) simplex.
- Mutating we get new (r-1)-simplex sharing (r-2)-face of the initial one.
- Glue them together. Do all possible mutations to obtain a polytope.

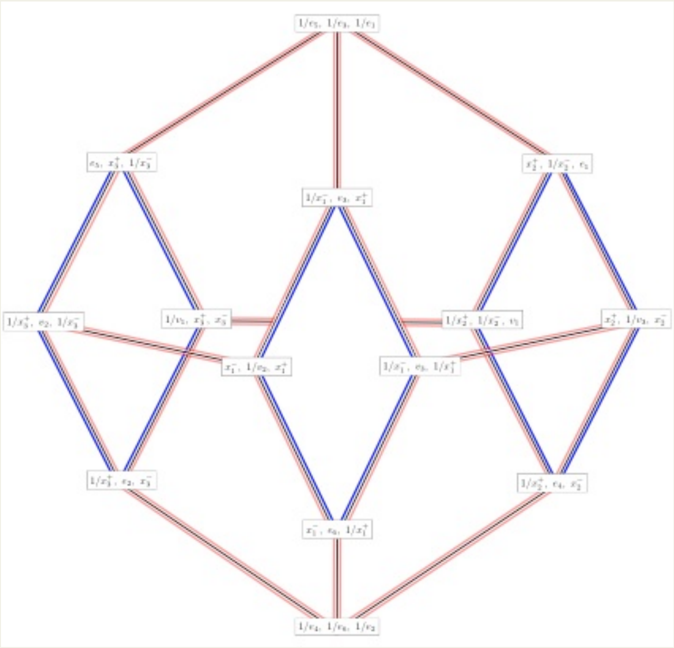
A_3

9 V, 21 E, 14 F



14 V, 21 E, 9 F

9F=3S+6P



E_6 cluster algebra

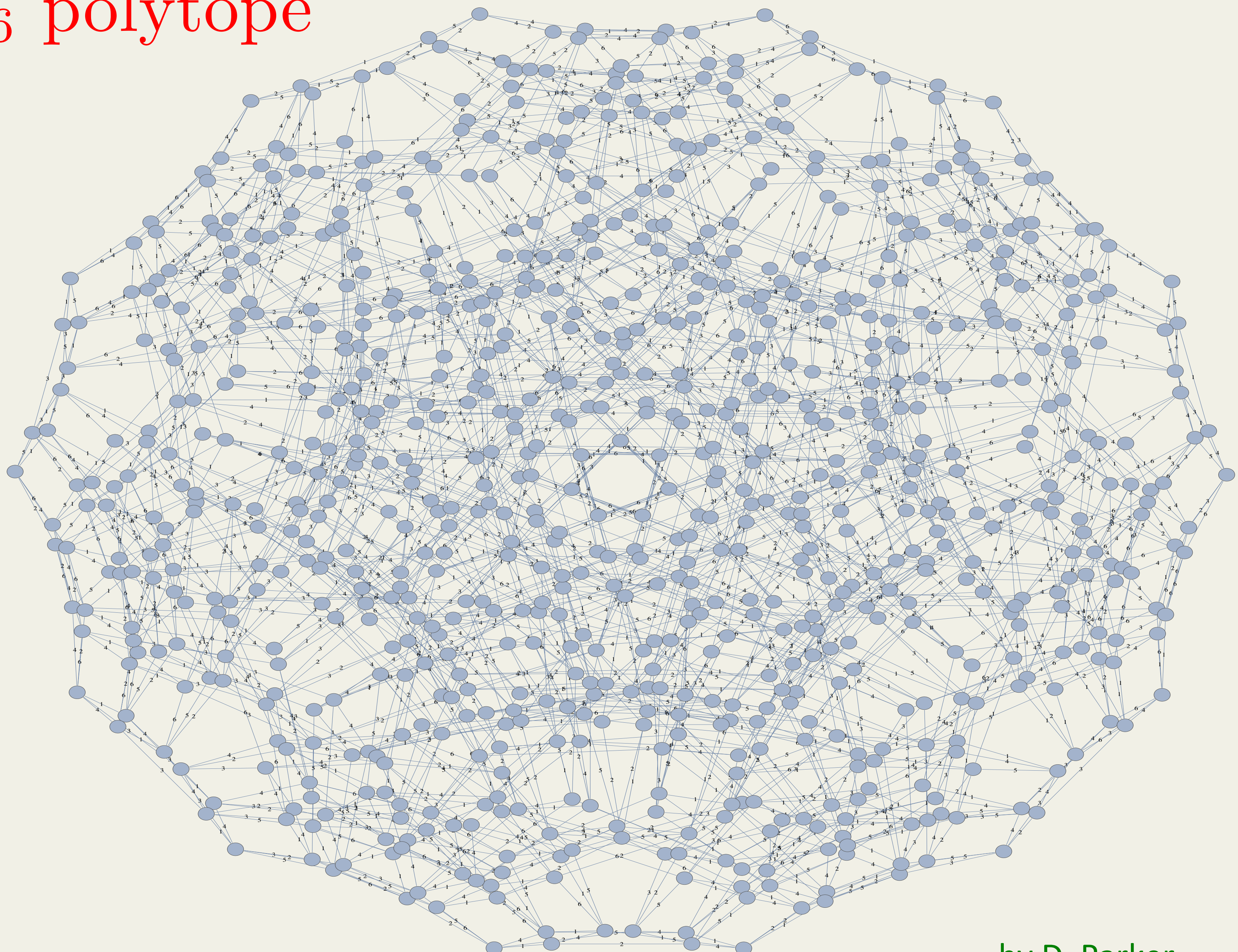
- 49 A-coordinates: 35 Plucker $\langle ijk \rangle$

  brackets $\langle 1\times 2, 3\times 4, 5\times 6\rangle$, $\langle 1\times 2, 3\times 4, 5\times 7\rangle$ + cyclic =14

  $$\langle 1\times 2, 3\times 4, 5\times 6\rangle = \langle 512\rangle\langle 634\rangle - \langle 534\rangle\langle 612\rangle$$

- Mutations generate 833 clusters
- E_6 Stasheff polytope: 833 V, 2499 E, 2856 F2 (1785S+ 1071P), 1547 F3
- Analyzing all quivers: 385 X-coordinates

E_6 polytope

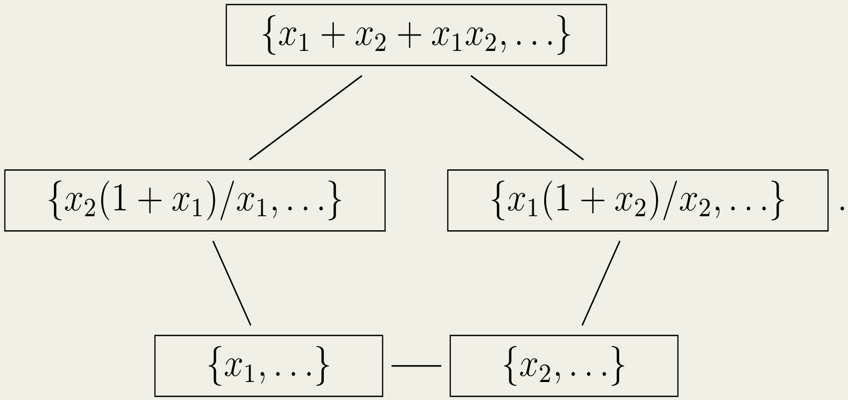


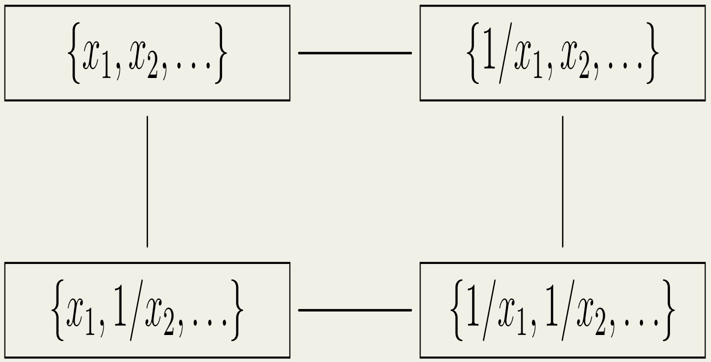
by D. Parker

Poisson bracket

- There is a natural Poisson bracket on X-coordinates in a given cluster: $\{X_i, X_j\} = b_{ij} X_i X_j.$
- It is invariant under mutations.
- Geometrically: coordinates have Poisson bracket

± 1 0





2-loop 7-point and E_6 Cluster Algebra

- All $\{x\}_2$ and $\{x\}_3$ in coproduct for 2-loop 7-points amplitude are cluster X-coordinates for E_6 cluster algebra
- Out of 385 only 231 appear in the amplitude.

  What is the criterion?? [Note: 9/15=231/385!]
- For each $\{x_1\}_2 \wedge \{x_2\}_2$ x_1 and x_2 are in the same cluster. Appear in pairs with zero Poisson bracket.
- $\Lambda^2 B_2$ is a sum of 42 squares of E_6 Stasheff polytope.

with Golden, Goncharov, Spradlin, Vergu

Cluster Polylogarithm

In order to find the corresponding function, we need to find a function whose coproduct can be expressed entirely in terms of cluster coordinates

$\mathrm{Li}_4(-x)$ ✓ $\mathrm{Li}_4(1+x)$ ✗

$$\delta \mathrm{Li}_{2,2}(x,y)|_{B_2\wedge B_2} = \{-y\}_2 \wedge \{-x\}_2 - \{-xy\}_2 \wedge \{-x\}_2 + \{-xy\}_2 \wedge \{-y\}_2,$$

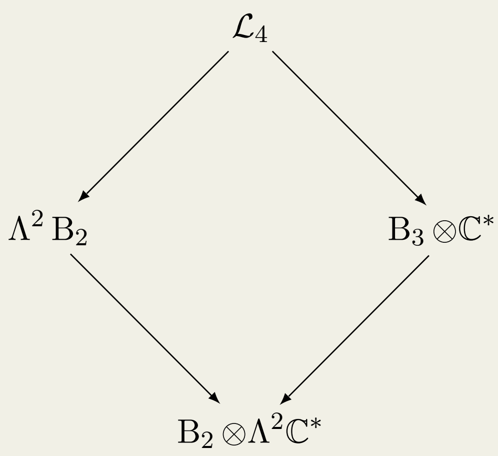
$$\begin{aligned}\delta \mathrm{Li}_{2,2}(x,y)|_{B_3\otimes\mathbb{C}^*} = &\{-x\}_3 \otimes y - 2\{-x\}_3 \otimes (xy-1) - \{-y\}_3 \otimes x + 2\{-y\}_3 \otimes (xy-1) \\ &- \left\{\frac{1-x}{xy-1}\right\}_3 \otimes x + \left\{\frac{1-y}{xy-1}\right\}_3 \otimes y - \{xy-1\}_3 \otimes y - \left\{\frac{xy}{1-xy}\right\}_3 \otimes x \\ &- \left\{\frac{x(1-y)}{xy-1}\right\}_3 \otimes y + \left\{\frac{(1-x)y}{xy-1}\right\}_3 \otimes x + \{x-1\}_3 \otimes x - \{y-1\}_3 \otimes y\end{aligned}$$

From coproduct to functions

Given $\quad b_{22} \in \Lambda^2 B_2 \quad$ and $\quad b_{31} \in B_3 \otimes C^*$

function f_4 with these coproduct components exists iff

$$\delta^2 f_4 = \delta(b_{22}) + \delta(b_{31}) = 0$$



$$\delta(\{x\}_2 \wedge \{y\}_2) = \{x\}_2 \bigotimes (1+y) \wedge y - \{y\}_2 \bigotimes (1+x) \wedge x$$

$$\delta(\{x\}_3 \otimes y) = \{x\}_2 \bigotimes x \wedge y.$$

A_2 Cluster Polylogarithms

Make an ansatz that the coproduct is a general linear combination of the available x-coordinates, and then solve:

$$\delta\left(\sum_{i,j}^{5} a_{ij}\{x_i\}_2 \wedge \{x_j\}_2 + b_{ij}\{x_i\}_3 \otimes x_j\right) = 0$$

There is a unique solution!

A_2 Cluster Polylogarithm

$$f_{A_2} \sim \sum_{i,j}^{5} j L_{2,2}(x_i, x_{i+j})$$

$$L_{2,2}(x,y) = \frac{1}{2}\mathrm{Li}_{2,2}\left(\frac{x}{y}, -y\right) + \frac{1}{6}\left(\mathrm{Li}_4\left(\frac{1+x}{xy}\right) + \mathrm{Li}_4\left(\frac{x(1+y)}{y(1+x)}\right)\right)$$
$$+ \frac{1}{5}\left(\mathrm{Li}_4\left(\frac{1+x}{xy}\right) + \frac{1}{2}\mathrm{Li}_4\left(\frac{1+x}{1+y}\right)\right) + \frac{1}{2}\mathrm{Li}_3\left(\frac{x}{y}\right)\log\left(\frac{1+x}{1+y}\right) - (x \leftrightarrow y)$$

$$\delta f_{A_2}(x_1, x_2)|_{\Lambda^2 \mathrm{B}_2} = \sum_{i,j=1}^{5} j\{x_i\}_2 \wedge \{x_{i+j}\}_2,$$

$$\delta f_{A_2}(x_1, x_2)|_{\mathrm{B}_3 \otimes \mathbb{C}^*} = 5\sum_{i=1}^{5}(\{x_{i+1}\}_3 \otimes x_i - \{x_i\}_3 \otimes x_{i+1})$$

A_3 Cluster Polylogarithm

Recall that $B_2 \wedge B_2$ derived from the amplitude side only has pairs with Poisson bracket zero, which is an additional constraint, and leads to a particular combination of pentagon functions.

A_3

$$x_1 \to x_2 \to x_3$$

$$\begin{aligned}
x_{1,1} &= x_1 & x_{1,2} &= 1/x_3 & v_1 &= \frac{(x_2+1)(x_1x_2x_3+x_2x_3+x_3+1)}{x_1x_2} \\
x_{2,1} &= (x_1x_2+x_2+1)\,x_3 & x_{2,2} &= \frac{x_1x_2+x_2+1}{x_1} & v_2 &= \frac{x_3+1}{x_2x_3} \\
x_{3,1} &= \frac{x_2x_3+x_3+1}{x_2} & x_{3,2} &= \frac{x_2x_3+x_3+1}{x_1x_2x_3} & v_3 &= (x_1+1)\,x_2 \qquad (4.1) \\
e_1 &= \frac{x_1x_2x_3+x_2x_3+x_3+1}{(x_1+1)\,x_2} & e_2 &= \frac{1}{(x_2+1)\,x_3} & e_3 &= \frac{(x_1+1)\,x_2x_3}{x_3+1} \\
e_4 &= \frac{x_2+1}{x_1x_2} & e_5 &= \frac{x_1(x_3+1)}{x_1x_2x_3+x_2x_3+x_3+1} & e_6 &= x_2.
\end{aligned}$$

$$\{x_{i,1}, x_{i,2}\} = 0, \quad \{e_i, e_{i+4}\} = 1, \quad \{v_i, x_{i\pm1,a}\} = \mp 1, \quad \{e_i, x_{i+1,a}\} = -1,$$

Cluster Polylogarithm

$$f_{A_3} = \frac{1}{2}\sum_{i=1}^{6}(-1)^i f_{A_2}(e_i, 1/e_{i+2})$$

$$\delta f_{A_3}|_{\Lambda^2 B_2} = \sum_{i=1}^{3}\{x_{i,1}\}_2 \wedge \{x_{i,2}\}_2$$

All non-trivial degree 4 cluster functions for E_6 are linear combinations of A_3 functions

D. Parker and A. Scherlis

2-loop 7-point amplitude

$$R_7^{(2)} = \frac{1}{2} f_{A_3}\left(\frac{\langle 1245\rangle\langle 1567\rangle}{\langle 1257\rangle\langle 1456\rangle}, \frac{\langle 1235\rangle\langle 1456\rangle}{\langle 1256\rangle\langle 1345\rangle}, \frac{\langle 1234\rangle\langle 1257\rangle}{\langle 1237\rangle\langle 1245\rangle}\right) + \frac{1}{2} f_{A_3}\left(\frac{\langle 1345\rangle\langle 1567\rangle}{\langle 1357\rangle\langle 1456\rangle}, \frac{\langle 1235\rangle\langle 3456\rangle}{\langle 1356\rangle\langle 2345\rangle}, \frac{\langle 1234\rangle\langle 1357\rangle}{\langle 1237\rangle\langle 1345\rangle}\right)$$
$$-\mathrm{Li}_4\left(-\frac{\langle 1234\rangle\langle 1256\rangle}{\langle 1236\rangle\langle 1245\rangle}\right) - \mathrm{Li}_4\left(-\frac{\langle 1234\rangle\langle 1257\rangle}{\langle 1237\rangle\langle 1245\rangle}\right) - \frac{1}{2}\mathrm{Li}_4\left(-\frac{\langle 1234\rangle\langle 1357\rangle}{\langle 1237\rangle\langle 1345\rangle}\right) - \frac{1}{2}\mathrm{Li}_4\left(-\frac{\langle 1234\rangle\langle 1456\rangle}{\langle 1246\rangle\langle 1345\rangle}\right)$$
$$+ \text{dihedral} + \text{parity conjugate} + \text{products of terms of lower weight.}$$

with Golden, Spradlin, Paulos

Cluster polylog functions are building blocks necessary to write down
all-n function for 2-loop MHV.

Conclusion

- We have advocated the study of cluster structure of N=4 YM amplitudes.
- We can use this structure for advancing computations.
- Many questions remain: cluster structure @ loops, other helicities, strong coupling…..

  very impressive explicit results by Dixon, Drummond, Duhr, Henn, Pennington, von Hippel & Basso, Sever Vieira
- Connection to the integrands: cluster structure also appeared in on-shell diagrams

  [Arkahi-Hamed, Bourjaily, Cachazo, Goncharov, Postnikov, Trnka]