Twistor Diagrams: ancient and modern

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New geometric structures in scattering amplitudes
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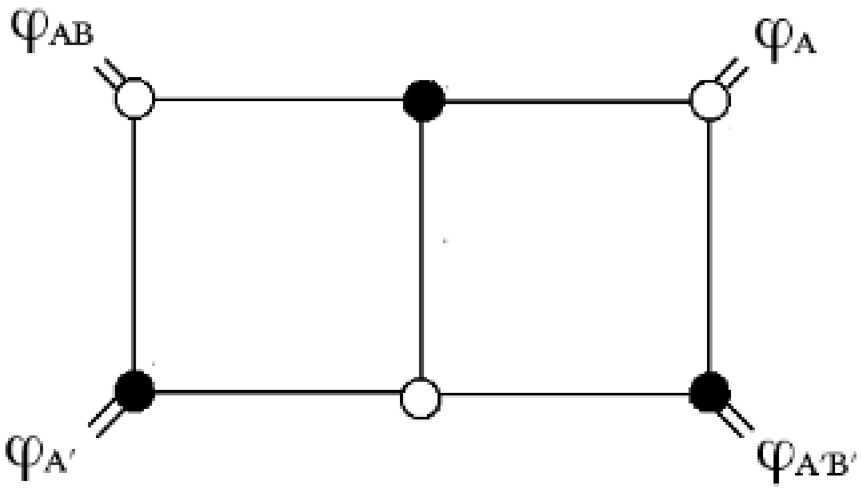
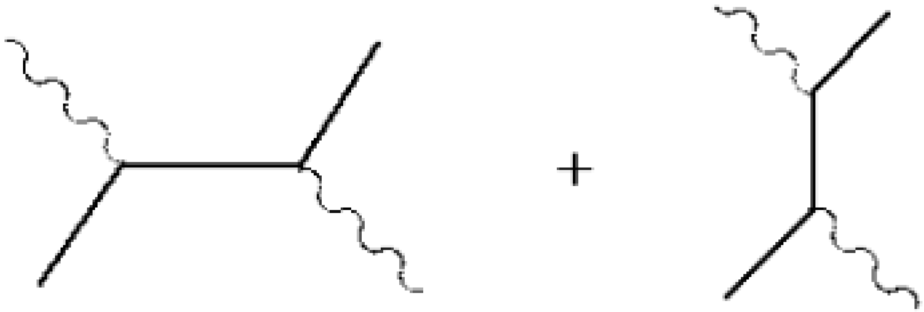
An ancient diagram

Penrose (1972):



for Compton scattering in massless QED:



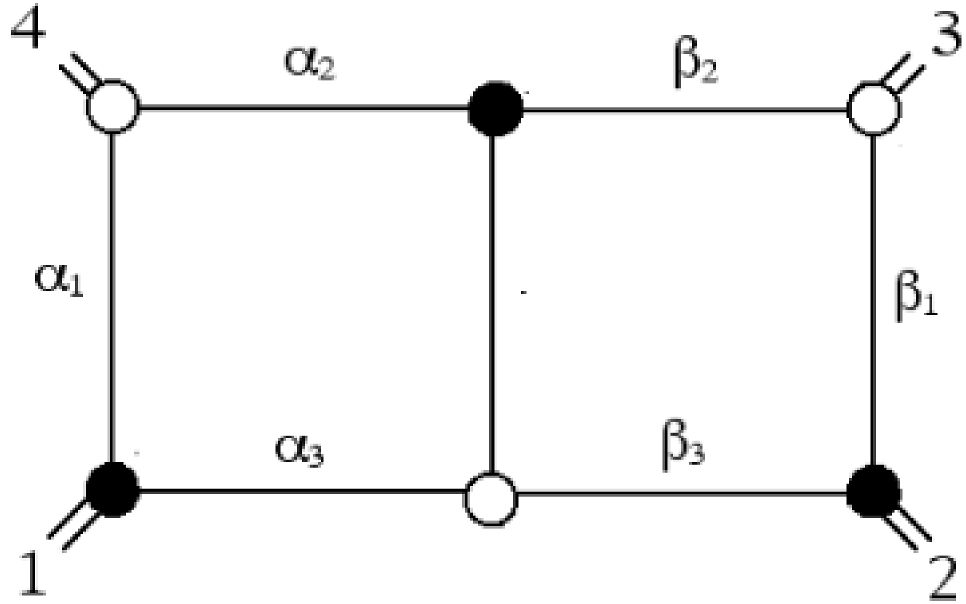
This diagram was conjectured by R.P. based on arguments from Hamiltonian structures.

How to verify? Two approaches:

(1) make a formal argument using external momentum eigenstates.

(2) consider as contour integral in (product of) twistor spaces, giving an exact functional of the in- and out-states.

The first approach led to the evaluation by looking at the more general diagram where the helicities have been ‘deformed’:



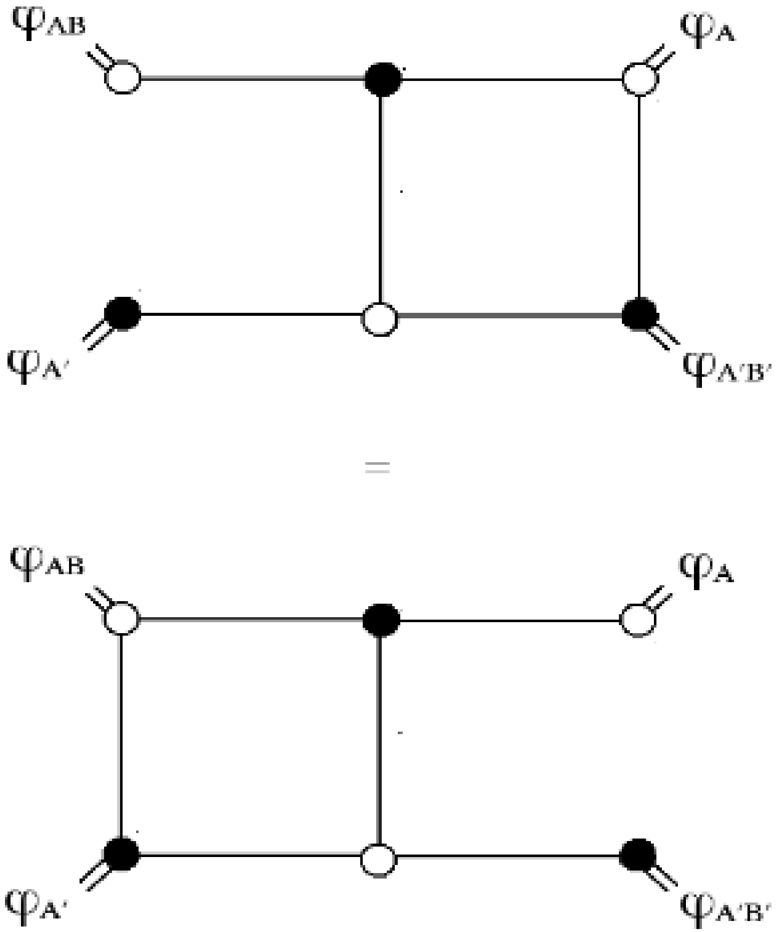
R.P. evaluated this as

$$\frac{\langle 12\rangle^2}{\langle 14\rangle\langle 34\rangle} \times \frac{\langle 34\rangle^{\alpha_1+\beta_1}\langle 23\rangle^{\alpha_2+\beta_2}\langle 14\rangle^{\alpha_2+\beta_3}}{\langle 12\rangle^{\alpha_2+\beta_2}\langle 13\rangle^{\alpha_1+\alpha_2}\langle 24\rangle^{\beta_1+\beta_3}} \times F\left(\frac{\langle 14\rangle\langle 23\rangle}{\langle 13\rangle\langle 24\rangle}\right) \times \delta\left(\sum p_i\right)$$

where F is a hypergeometric function. In the limit where the parameters vanish, F =1, and the simple MHV formula remains.

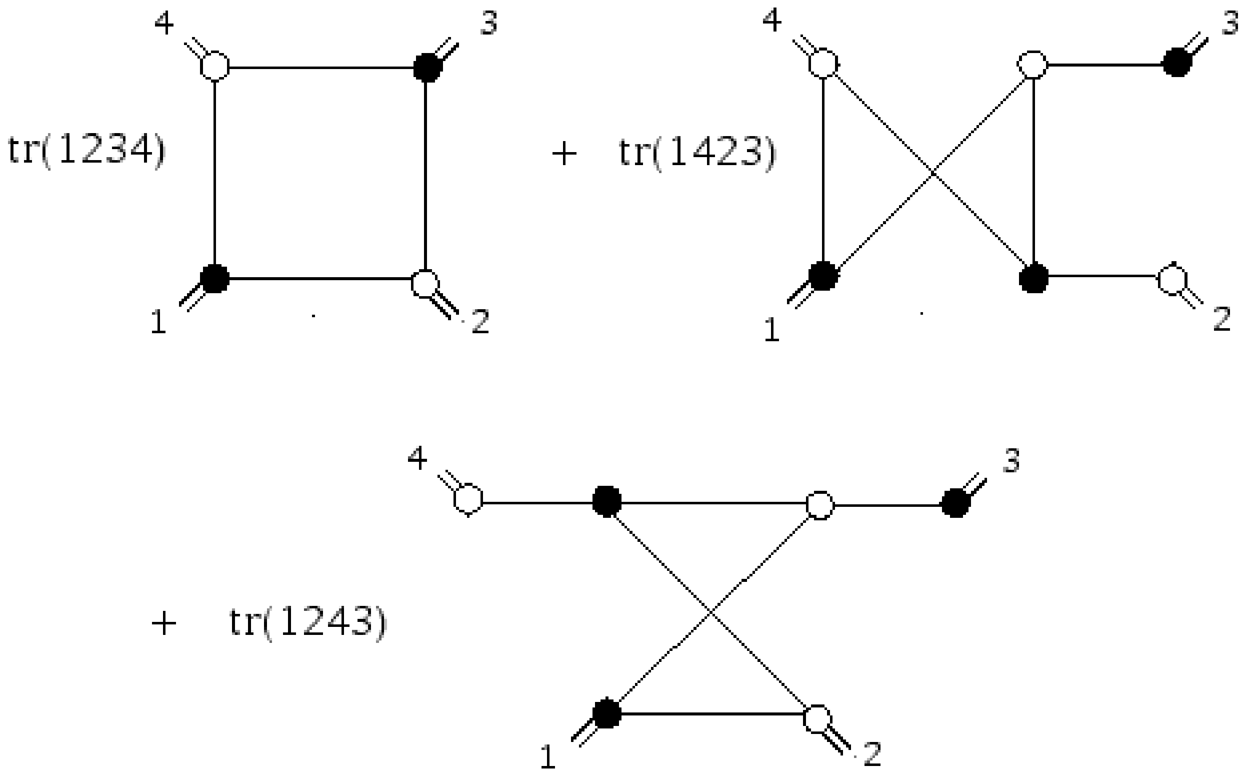
A further connection with modern developments:
the *non-uniqueness* of the 1972 diagram.

The Compton scattering amplitude also arises equally well from the less symmetric but simpler single-box diagrams: thus



A puzzle in the old diagram theory was to find a rule or generating principle that would explain this.

By 1990: the non-unique structure was found in all the massless field interactions that had been studied. The most fundamental case was that of *pure (non-Abelian) gauge-field* interaction. Here (AH, 1990)



but with the same question about uniqueness of representation.

From twistor diagrams to modern on-shell diagrams

Extension of results to double Compton scattering (i.e. 5 fields) did not go far enough to see the extension of the MHV pattern. Parke-Taylor not known!

But after Witten (2003), all this material enabled the recgnition of BCFW as a joining of twistor diagrams (AH 2005), with natural extension to super-BCFW in N=4 SYM.

Hence twistor diagrams for all tree amplitudes in N=4 SYM.

The *square identity* noted as the feature of the diagram theory (generalises the equivalence noted for the two single-box representations of the Compton scattering amplitude above).

The evaluation of general diagrams in terms of momentum states now becomes a significant combinatorial problem. Solved by Nima Arkani-Hamed, Freddy Cachazo and group, by Grassmannian formalism (2008). Extended beyond tree amplitudes to leading singularities and then to loop integrands.

Theory of positive Grassmannian (2012) identifies cells with on-shell diagrams — essentially the same diagrams as in R.P's original paper, but with a new interpretation.

Diagrams now are elements of a well-defined graph theory, with equivalence classes defined precisely by the square identity.

The double-box diagram now is interpreted as a distinct object, with a *reduction process* showing why it is equivalent to the single boxes.

Four twistor-diagram loops give a single Feynman loop.

The integration theory not as well-defined as the theory of tree amplitudes and of loop integrands.

So the ancient diagram is a reminder of where more insight is needed.

Renewed interest in conformally invariant deformation of these diagrams (T.Lukowski talk)
to study regularisation — now within a much larger and clearly defined theoretical framework.

But perhaps ancient ideas might have some relevance!

Diagrams as twistor integrals.

Also pursued by R.P. from the start. Natural idea of summing over all possibilities.

More fundamental since purely twistorial, but also intractable!

Inspiration: the no-interaction formula, i.e. inner product of in and out states.

Useful notation:

$$[x]_r = -\frac{\Gamma(r+1)}{(-x)^{r+1}}$$

so that $[x]_n$ behaves like the n^{th} derivative of a delta-function.
This allows the definition of

$$[W.Z]_r = -\frac{\Gamma(r+1)}{(-W.Z)^{r+1}}$$

Thus

$$\frac{\partial}{\partial W_\alpha}[W.Z]_r = Z^\alpha [W.Z]_{r+1}$$

The inner product, for helicity n, is then given by a compact (6-dim) contour integral

$$\oint f(Z^\alpha) g(W_\alpha) \, [W.Z]_n \, DZ \wedge DW$$

The idea was to extend such compact contour integrals to the evaluation of twistor diagrams, again using genuine finite-normed in- and out-states, expressed as twistor 1-functions, on the outside. The results would then be manifestly conformally invariant and manifestly finite.

Problems:

(1) Defining the fundamental line $[W.Z]_{-1}$ ... an anti-derivative.

One approach: define this as a *boundary* on W.Z=0. For the Coulomb scattering diagram, this was valid. In general, the appropriate boundary contours do not exist. The reason lies in the infra-red divergences associated with forward-direction scattering. This was seen through the other approach: define $[W.Z]_{-1}$ as the limit of $[W.Z]_{-1+\alpha}$ for α a complex parameter, and see the pole as α —> 0.

Boundaries lead to amplitudes as *volumes*... (and the later polytope picture).
Relative homology (S. Huggett and others).

(2) Finding contours for each channel

(3) Expressing the content of crossing symmetry (without complex momenta).

The description of contours for diagrams with complex powers was considerably developed. The evaluation of the double box was studied and R.P's hypergeometric function justified and refined. (AH, O'Donald, Müller, in 1990s).

Possibly this may be relevant to the 'deformed' diagrams now being studied.

Illustration of compact contour integration

The classic Pochhammer contour can be written more symmetrically in CP^1

$$\frac{1}{(2\pi i)^2}\oint (x.\alpha)^{-a}(x.\beta)^{-b}(x.\gamma)^{-c}Dx = \frac{(\alpha.\beta)^{c-1}(\beta.\gamma)^{a-1}(\gamma.\alpha)^{b-1}}{\Gamma(a)\,\Gamma(b)\,\Gamma(c)}$$

where a + b + c = 2.

This can be written using the [] notation as

$$\frac{1}{(2\pi i)^2}\oint [x.\alpha]_{a-1}[x.\beta]_{b-1}[x.\gamma]_{c-1}Dx = (\alpha.\beta)^{c-1}(\beta.\gamma)^{a-1}(\gamma.\alpha)^{b-1}$$

Not so familiar, in CP^2, for a + b + c + d = 3,

$$\frac{1}{(2\pi i)^3}\oint [x.\alpha]_{a-1}[x.\beta]_{b-1}[x.\gamma]_{c-1}[x.\delta]_{d-1}D^2x$$
$$= \langle\alpha\beta\gamma\rangle^{d-1}\langle\beta\gamma\delta\rangle^{a-1}\langle\gamma\delta\alpha\rangle^{b-1}\langle\delta\alpha\beta\rangle^{c-1}$$

where the analogous contour has the topology of an S^2 .

In the case d=3, and taking the limit a, b, c —> 0, the area of a triangle:

$$\frac{1}{(2\pi i)^3}\oint \frac{[x.\alpha]_{-1}[x.\beta]_{-1}[x.\gamma]_{-1}}{(x.\delta)^3} D^2 x = \frac{1}{2}\frac{\langle\alpha\beta\gamma\rangle^2}{\langle\beta\gamma\delta\rangle\langle\gamma\delta\alpha\rangle\langle\delta\alpha\beta\rangle}$$

This shows a way to define a deformed area.

The formula generalises to all n in the natural way.

Relevant to the 4-simplex and then the cyclic polytopes for NMHV amplitudes in momentum-twistor space.

Relations between different orderings

Modern diagram theory has the colour traces attached to ring-ordered partial amplitudes.

Applying this in the case of U(1), i.e. to massless QED, shows that the original Compton scattering diagram only gave the correct answer because of the U(1) decoupling identity:

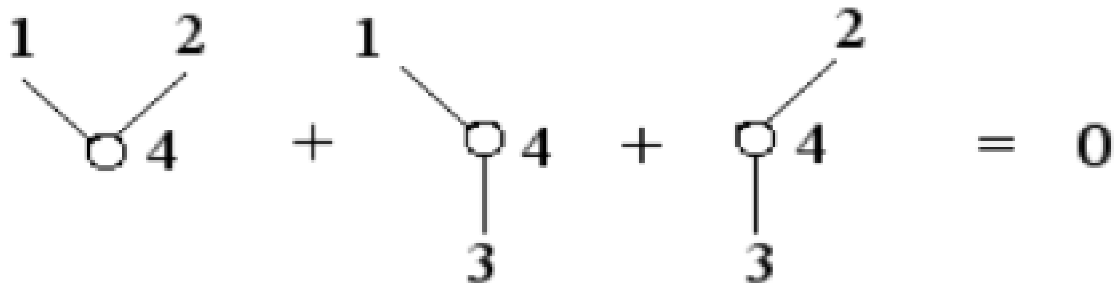
$$A(1234) + A(1342) + A(1423) = 0.$$

Striking example of this: for n photons, the amplitude is 0.

But the method using momentum-twistors means a summation of $(n-1)!/2$ terms to 0.

There must be a simple geometrical structure which combines different orderings and makes this apparent.

Within twistor diagrams for MHV amplitudes, this can be provided by the contour relation



This extends usefully to gravitational scatterings (AH 2011). Many relations between diagrams.

Recent developments (CHY...) suggest there should be further connections between

— amplitudes-as-volumes

— colour factors and kinematic numerators participating in the geometry.

Concluding remarks

The original twistor diagram theory did not achieve its original purpose of obtaining amplitudes as functionals of the genuine Fock spaces of states, in a manifestly finite and conformally invarant way.

But it led to the theory which has enormously extended the theory of amplitudes as functions of momenta.

It also suggests the possibilities of

(1) Obtaining amplitudes as pure (super-)volume.

(2) Obtaining amplitudes in an entirely twistor-geometric form which does *not* appeal to external momentum states

These ideas might be combined by the use of *elemental* states (talk by Y. Geyer) as external fields (states supported on a point, or plane, in projective twistor space). These seem to have correct crossing properties. But their interpretation for Minkowski space unclear.

Interesting relation with standard wave-functions: a spin-1 field can naturally be obtained as the integral of an elemental state round a *closed curve.*

With such external states the entire amplitude becomes a (super-)volume.

More speculative idea from the old theory (1985): twistor *scale* could provide a new method for regularisation, without going beyond four dimensions.

More generally: R.P's idea was to that twistor geometry could suggest completely new structures for quantum fields in curved space-time... (next talk!)