### Amplitude as a flux tube gas : Scalars

Benjamin Basso ENS Paris

#### New geometric structures in scattering amplitudes Oxford

#### based on work with Amit Sever and Pedro Vieira

# Wilson loops at finite coupling in N=4 SYM

[Alday, Gaiotto, Maldacena, Sever, Vieira'10]



I + I d background : *flux tube* sourced by two parallel null lines bottom&top cap excite the *flux tube* out of its ground state

Sum over all flux-tube eigenstates

$$\mathcal{W} = \sum_{\text{states } \psi} C_{\text{bot}}(\psi) \times e^{-E(\psi)\tau + ip(\psi)\sigma + im(\psi)\phi} \times C_{\text{top}}(\psi)$$

### **Refinement : the pentagon way**

[BB,Sever,Vieira'13]



Valid at any coupling

$$= \sum_{\psi_i} \left[ \prod_i e^{-E_i \tau_i + ip_i \sigma_i + im_i \phi_i} \right] \times$$

 $P(0|\psi_1)P(\psi_1|\psi_2)P(\psi_2|\psi_3)P(\psi_3|0)$ 

# **Refinement : the pentagon way**

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To compute amplitudes we need

The spectrum of flux-tube states  $~\psi$ 

All the pentagon transitions  $\ P(\psi_1|\psi_2)$ 

### **Beyond the area paradigm**

[Alday, Maldacena'07]



#### classical

 $\mathcal{W}_{n=6} = f_6 \,\lambda^{-\frac{7}{288}} e^{\frac{\sqrt{\lambda}}{144} - \frac{\sqrt{\lambda}}{2\pi}A_{n=6}} (1 + O(1/\sqrt{\lambda}))$ 

# $\begin{array}{c} \textbf{minimal area}\\ \textbf{in}\\ AdS_5 \end{array}$

quantum

[Alday, Gaiotto, Maldacena'09] [Alday, Maldacena, Sever, Vieira'10]

**Pre-factor** 

$$f_6 = \frac{1.04}{(\sigma^2 + \tau^2)^{1/72}} + O(e^{-\sqrt{2}\tau})$$

[BB,Sever,Vieira'14]

#### The flux-tube eigenstates



#### **Spectral data**

$$E = E(u_1) + E(u_2) + \ldots + E(u_N)$$
rapidity

$$(u) = \text{twist} + a^2$$
.

$$p = p(u_1) + \dots + p(u_N)$$

$$p(u) = 2u + g^2.$$

#### can be found using integrability

E(

#### Pentagon/OPE series for hexagon

$$\mathcal{W}_{\text{hex}} = \int \int d\mathbf{u} P_{\mathbf{a}}(0|\mathbf{u}) e^{-E(\mathbf{u})\tau + ip(\mathbf{u})\sigma + im\phi} P_{\mathbf{a}}(\bar{\mathbf{u}}|0)$$

i.e. in collinear limit

#### Lightest states dominate at large au

What are they?



For  $\tau \gg 1$  all heavy flux tube excitations decouple

Low energy effective theory : (relativistic) O(6) sigma model

[Alday, Maldacena'07]

$$\mathcal{L}_{\text{eff}} = \frac{\sqrt{\lambda}}{4\pi} \,\partial X \cdot \partial X \qquad \text{with} \qquad X^2 = \sum_{i=1}^6 X_i^2 = 1$$

# Minimal surface from low-energy viewpoint



# Minimal surface from low-energy viewpoint



# Minimal surface from low-energy viewpoint





One can go around the pentagon with 5 mirror rotations



This is one more than for a square

$$E \xrightarrow{\gamma} ip \longrightarrow -E \longrightarrow -ip \longrightarrow E$$

#### Pentagon as twist operator

In short, a pentagon = 5 quadrants glued together



#### Hexagon as a correlator of twist operators



 $\mathcal{W}_{6} = \langle 0 | \phi_{\bigcirc}(\tau, \sigma) \phi_{\bigcirc}(0, 0) | 0 \rangle$ 

#### Hexagon as a correlator of twist operators



corrections from heavy modes irrelevant in collinear limit

$$\mathcal{W}_6 = \langle 0 | \phi_{\bigcirc}(\tau, \sigma) \phi_{\bigcirc}(0, 0) | 0 \rangle + O(e^{-\sqrt{2\tau}})$$

Probes the physics of the O(6) sigma model :

 $\mathcal{W}_{O(6)}(z)$ 

 $z = m\sqrt{\sigma^2 + \tau^2}$ 

Large distance $z \gg 1$  $\mathcal{W}_{O(6)} = 1 + O(e^{-2z})$ Short distance $z \ll 1$  $\mathcal{W}_{O(6)} = ?$ 

# **OPE** as form factor expansion

Insert complete basis of states

See [Cardy,Castro-Alvaredo,Doyon'07] for similar considerations for computing entanglement entropy in integrable QFT

$$\mathcal{W}_{O(6)} = \sum_{N} \frac{1}{N!} \int \frac{d\theta_1 \dots d\theta_N}{(2\pi)^N} \langle 0 | \phi_{\bigcirc} | \theta_1, \dots, \theta_N \rangle \langle \theta_1, \dots, \theta_N | \phi_{\bigcirc} | 0 \rangle e^{-z \sum_{i} \cosh \theta_i}$$

#### Pentagon transition = form factor of twist operator

$$P(0|\theta_1,\ldots,\theta_N) = \langle \theta_1,\ldots,\theta_N | \phi_{\bigcirc} | 0 \rangle$$

#### Normalization

#### $\langle 0 | \phi_{\bigcirc} | 0 \rangle = 1$ which enforces that $\mathcal{W}_{O(6)} \to 1$ $z \to \infty$

$$\mathcal{W}_{6} = 1 + \frac{1}{2} \int \frac{d\theta_{1} d\theta_{2}}{(2\pi)^{2}} |P(0|\theta_{1}, \theta_{2})|^{2} e^{-m\tau(\cosh\theta_{1} + \cosh\theta_{2}) + im\sigma(\sinh\theta_{1} + \sinh\theta_{2})} + \dots$$
**two-particle form factor**

$$|P(0|\theta_1, \theta_2)|^2 = \mu^2 F(\theta_1 - \theta_2)$$

#### where

$$F(\theta) = \frac{8\theta \tanh\left(\frac{\theta}{2}\right)\Gamma\left(\frac{3}{4} + \frac{i\theta}{2\pi}\right)\Gamma\left(\frac{3}{4} - \frac{i\theta}{2\pi}\right)}{\pi\Gamma\left(\frac{1}{4} + \frac{i\theta}{2\pi}\right)\Gamma\left(\frac{1}{4} - \frac{i\theta}{2\pi}\right)} \qquad \mu = \frac{2\Gamma(\frac{3}{4})}{\sqrt{\pi}\Gamma(\frac{1}{4})}$$

$$\mathcal{W}_6 = 1 + \frac{1}{2} \int \frac{d\theta_1 d\theta_2}{(2\pi)^2} |P(0|\theta_1, \theta_2)|^2 e^{-m\tau(\cosh\theta_1 + \cosh\theta_2) + im\sigma(\sinh\theta_1 + \sinh\theta_2)} + \dots$$

#### multi-particle states

Multi-particle transitions



$$\mathcal{W}_6 = 1 + \frac{1}{2} \int \frac{d\theta_1 d\theta_2}{(2\pi)^2} |P(0|\theta_1, \theta_2)|^2 e^{-m\tau(\cosh\theta_1 + \cosh\theta_2) + im\sigma(\sinh\theta_1 + \sinh\theta_2)} + \dots$$

#### multi-particle states



# Interlude on matrix part

Integral representation of rational part

$$r = \frac{1}{K_1!K_2!K_3!} \int \prod_i \frac{dw_{1,i}}{2\pi} \prod_i \frac{dw_{2,i}}{2\pi} \prod_i \frac{dw_{3,i}}{2\pi}$$

$$\times \frac{\prod_{i < j} g(w_{1,i} - w_{1,i}) \prod_{i < j} g(w_{2,i} - w_{2,i}) \prod_{i < j} g(w_{3,i} - w_{3,i})}{\prod_{i,j} f(w_{2,i} - \frac{2}{\pi} \theta_j) \prod_{i,j} f(w_{1,i} - w_{2,j}) \prod_{i,j} f(w_{3,i} - w_{2,j})}$$

 $\theta$ 

 $w_{2}$ 

$$g(x) = x^2(x^2 + 1)$$
  $f(x) = x^2 + \frac{1}{4}$ 

Here (for singlet states)  $K_2 = K_\theta$   $K_1 = K_3 = \frac{1}{2}K_\theta$ 

$$\mathcal{W}_6 = 1 + \frac{1}{2} \int \frac{d\theta_1 d\theta_2}{(2\pi)^2} |P(0|\theta_1, \theta_2)|^2 e^{-m\tau(\cosh\theta_1 + \cosh\theta_2) + im\sigma(\sinh\theta_1 + \sinh\theta_2)} + \dots$$

#### multi-particle states

Multi-particle transitions  $\theta_1 \theta_2 \theta_3 \theta_4 = (\pi_1 + \pi_2) + (\pi_3 + \pi_3)$ 

General formula:

We have all ingredients! We can plot the form factor series...

θ

integrand = 
$$\prod_{i < j} \frac{1}{P(\theta_i | \theta_j) P(\theta_j | \theta_i)} \times \text{rational}$$

# Numerical analysis



# Plot of the truncated OPE/form factor series representation for $\log W_{O(6)}$

Short distance OPE (valid for  $z \ll 1$ )

$$\phi_{\bigcirc}(\tau,\sigma)\phi_{\bigcirc}(0,0) \sim \frac{\log\left(1/z\right)^{B}}{z^{A}}\phi_{\bigcirc}(0,0)$$
3-point function

Critical exponent A

$$A = 2\Delta_{\bigcirc} - \Delta_{\bigcirc} = 2\Delta_{5/4} - \Delta_{3/2}$$

with  $\Delta_k$  the scaling dimension of the twist operator  $\phi_k$ 

[Knizhnik'87] [Lunin,Mathur'00] [Calabrese,Cardy'04]

$$\Delta_k = \frac{c}{12}\left(k - \frac{1}{k}\right)$$

$$c = \text{central charge}$$
  
 $2\pi(k-1) = \text{excess angle for } \phi_k$ 

Short distance OPE (valid for  $z \ll 1$ )

$$\phi_{\bigcirc}(\tau,\sigma)\phi_{\bigcirc}(0,0) \sim \frac{\log\left(1/z\right)^{B}}{z^{A}}\phi_{\bigcirc}(0,0)$$
3-point function

Critical exponent A

$$A = \frac{1}{36} \qquad \text{since in our case } c = 5$$

Critical exponent B from one-loop anomalous dimensions

$$B = -\frac{3}{2}A = -\frac{1}{24}$$



#### Constant C is fixed in the IR by

 $\mathcal{W}_{O(6)} \to 1$  when  $z \to \infty$ 

#### and is thus non perturbative

# Numerical analysis



**Pre-factor**  $f_6 = \frac{1.04}{(\sigma^2 + \tau^2)^{1/72}} + O(e^{-\sqrt{2}\tau})$ 

# Infrared/non-perturbative regime



 $z \gg 1$  equivalently  $\tau \gg e^{\sqrt{\lambda}/4}$ 

Deep (infrared) collinear limit

**Completely non perturbative** 

#### **Cross over**



 $z \ll 1$  equivalently  $1 \ll \tau \ll e^{\sqrt{\lambda}/4}$ 

UV regime of O(6) model : *perturbative* collinear limit

#### **Cross over**



#### here we could match O(6) analysis with string perturbative expansion

# Full stringy pre-factor



*full thing :* include all heavy modes *gluons, fermions, ...* 

 $f_6 = \frac{1.04}{(\sigma^2 + \tau^2)^{1/72}} + O(e^{-\sqrt{2}\tau}) + O(e^{-2\tau})$ 

# **NMHV** amplitude I

Need so-called charged pentagon transitions (Pedro's talk)

Simplest case : pentagon carrying a vector index under SU(4)

$$\phi_{\widehat{\Box}}(\sigma,\tau) \to \phi_{\widehat{\Box}_{i}}(\sigma,\tau) \propto X_{i}(\sigma,\tau)\phi_{\widehat{\Box}}(\sigma,\tau)$$

NMHV hexagon (scalar component)

 $\langle 0 | \phi_{\widehat{O}_{i}}(\sigma, \tau) \phi_{\widehat{O}_{j}}(0, 0) | 0 \rangle = \delta_{ij} \times \mathcal{W}_{\text{NMHV}}(z)$ 

# **NMHV** amplitude II



UV behavior

$$\phi_{\widehat{\Box}_i}(\tau,\sigma)\phi_{\widehat{\Box}_j}(0,0) \sim \delta_{ij} \times \frac{\log\left(1/z\right)^{B'}}{z^A}\phi_{\bigcirc}(0,0)$$



# **NMHV** ratio function at strong coupling

$$R = \frac{\mathcal{W}_{\rm NMHV}}{\mathcal{W}_{\rm NMH}} = \frac{8}{\sqrt{\lambda}} \times \text{pure number} \times \log\left(1/z\right)$$



# **Higher-point amplitude**

Higher-point amplitudes correspond to higher-points correlators

$$\mathcal{W}_n = \langle 0 | \phi_{\bigcirc}(\tau_{n-4}, \sigma_{n-4}) \dots \phi_{\bigcirc}(\tau_1, \sigma_1) | 0 \rangle$$

could also be charged

Overall short-distance scaling is controlled by OPE

$$\underbrace{\phi_{\bigcirc} \dots \phi_{\bigcirc}}_{n-4} \sim m^{-(n-4)\Delta(\frac{5}{4}) + \Delta(\frac{n}{4})} \phi_{\varphi}$$

with final excess angle  $\varphi = 2\pi \times \frac{n-4}{4}$ 

This leads to the addition

$$\mathcal{W}_n = e^{-\frac{\sqrt{\lambda}}{2\pi}A_n + \frac{\sqrt{\lambda}(n-4)(n-5)}{48n} + o(\sqrt{\lambda})}$$

#### Conclusions

At strong coupling SA develop a non-perturbative regime in the near collinear limit

The string  $\alpha'$  expansion breaks down for extremely large values of  $\tau \sim -\log u_2 \sim e^{\sqrt{\lambda}/4}$ 

That's because flux tube mass gap m becomes extremely small

One should think in terms of correlators of twist operators

This fixes the collinear limit of SA at strong coupling



#### Full one-loop pre-factor including heavy modes?

#### Prefactor for higher-point amplitude /ratio function?

#### **One-loop Thermodynamical-Bubble-Ansatz equations?**

... and many other questions ...

# THANK YOU!