Walter Burke Institute for Theoretical Physics

New structures in non-planar $\mathcal{N} = 4$ SYM amplitudes

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N. Arkani-Hamed, J. Bourjaily, F. Cachazo, A. Postnikov, JT, to appear

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Why to study on-shell scattering amplitudes?

- Efficient predictions for colliders.
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- Four dimensional interacting QFT.
- Yangian symmetry $\rightarrow$ integrable.
- Interesting connections: twistor string, Wilson loop/amplitude correspondence $\rightarrow$ very powerful computational methods like flux tube S-matrix, amplitudes at finite coupling.

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Very different point of view: make all symmetries and properties of the amplitude **manifest**.

Dual formulation for planar amplitudes

[Arkani-Hamed, Bourjaily, Cachazo, Goncharov, Postnikov, JT, 2012]

Integrand

In the planar theory we can define **Integrand**.

- Gauge invariant rational function to be integrated.
- We can define it as a sum of Feynman diagrams prior to integration (using dual coordinates) or better as a function that satisfies all cut conditions.

$$A_n = \int d^4\ell_1\, d^4\ell_2 \dots d^4\ell_L\; \mathcal{I}_n(\ell_i, p_j)$$

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Why is this object interesting?

- Well-defined and finite (no IR divergencies, no regulators).
- Fascinating connections to recent discoveries in algebraic geometry and combinatorics.
- For this object we are able to find a completely new formulation – does it exist for integrated amplitudes?

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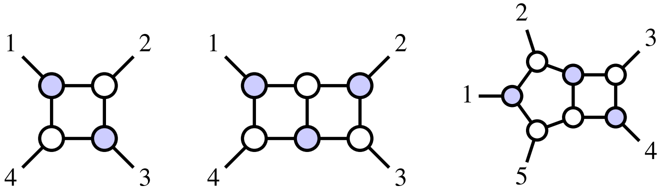
Searching for new expansion for planar $\mathcal{N} = 4$ SYM:

1. Traditional on-shell approach: using on-shell data to fix the amplitude. We can go further: define fully on-shell objects which directly serve as building blocks for the amplitude.
2. Yangian symmetry is obscured in the traditional formulation. New expansion should make it manifest term-by-term.

Dual formulation

The answer: **On-shell diagrams**.

- Well-defined object in any weakly coupled QFT: on-shell gluing of elementary amplitudes.
- In Yang-Mills theory we have two elementary 3pt amplitudes: white and black vertices



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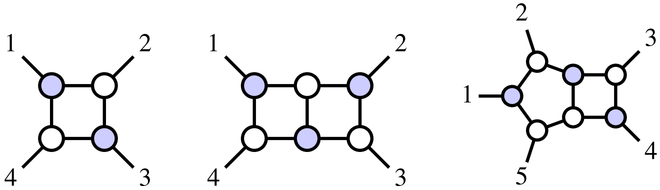

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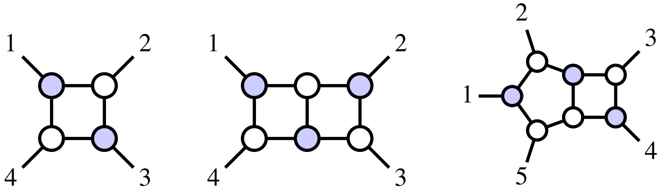

- We can use BCFW recursion relations to write the amplitude as a sum of on-shell diagrams.
- These diagrams are not local in spacetime: presence of spurious poles (like in BCFW).

Dual formulation

There is a completely different way how to look at these diagrams: relation to cells of Positive Grassmannian $G_+(k,n)$.

- $G_+(k,n)$: $(k \times n)$ matrix mod $GL(k)$

$$C = \begin{pmatrix} * & * & \dots & * \\ \vdots & \vdots & \dots & \vdots \\ * & * & \dots & * \end{pmatrix}$$

  where all maximal minors are positive, $(a_{i_1} a_{i_2} \dots a_{i_k}) > 0$.
- Stratification: cell of $G_+(k,n)$ of dimensionality d given by a set of constraints on consecutive minors.
- For each cell of dimensionality d we can find d positive coordinates x_i, and associate a logarithmic form

$$\Omega_0 = \int \frac{dx_1}{x_1} \dots \frac{dx_d}{x_d} \delta(C(x_i) Z_j)$$

Dual formulation

Further step: **Amplituhedron**

[Arkani-Hamed, JT, 2013]

- Glue pieces of the amplitude together and find a new definition of the amplitude as a single object with all symmetries and properties manifest.

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If all this is a consequence of integrability we loose everything one step away from planar $\mathcal{N} = 4$ SYM and we should not find anything special there.

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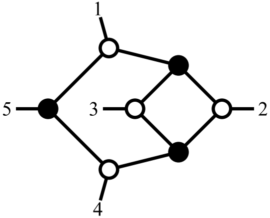
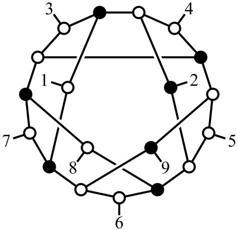
Before looking at amplitudes we can study on-shell diagrams. They are well-defined for non-planar case.





Non-planar on-shell diagrams

[Arkani-Hamed, Bourjaily, Cachazo, Postnikov, JT, to appear]

Non-planar on-shell diagrams

We can associate a cell in $G(k,n)$ and the logarithmic form which gives the same result as on-shell gluing.

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The amplitude is fully determined by its cuts so in the end we should be able to reverse the process and write the amplitude in terms of on-shell diagrams.

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We do not know how to do it now. Studying non-planar on-shell diagrams seems like a right step towards that goal.

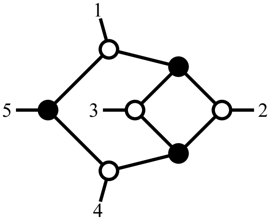
There are special properties of certain on-shell diagrams which do not follow from any known symmetries of $\mathcal{N}=4$ SYM.

Non-planar on-shell diagrams

We consider $k = 2$ on-shell diagrams relevant for MHV amplitudes.

We consider reduced diagrams

- No internal bubbles in the diagram (no unfixed parameters).
- Number of propagators equals to $4L$: the diagram is represented by rational function.
- We often refer to them as *leading singularities* as they represent $4L$ cuts of loop amplitudes.



Claim for MHV leading singularities

The statement for MHV leading singularities

- In planar sector we can get only a tree-level amplitude

$$A_n = \frac{1}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\ldots\langle n1\rangle}$$

  we refer to it as Parke-Taylor factor $P(123\ldots n)$.

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- Superconformal invariance: holomorphic function of λ only.

Our claim:

MHV leading singularities are linear combination of Parke-Taylor factors with different orderings $P(\sigma)$ and $+1$ coefficients.

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2. Even expressions with local poles might not be expressible in terms of Parke-Taylor factors, e.g.

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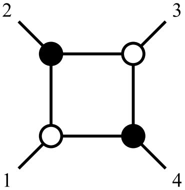
But this does not happen and we can indeed prove that the claim is correct.

Examples

Example 1: One-loop box



$$= \frac{1}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}$$

$$= P(1234)$$

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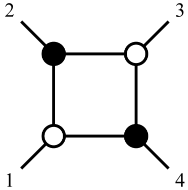

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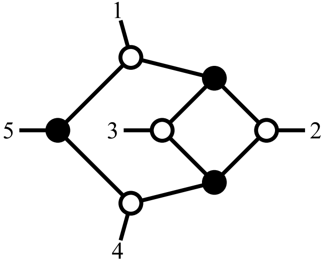
Example 2: Inverse soft-factor diagram

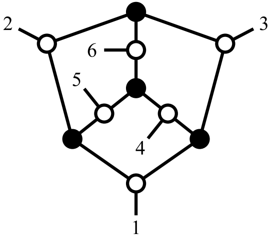


$$= \frac{\langle 41\rangle}{\langle 45\rangle\langle 51\rangle} \cdot \frac{1}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}$$

$$= P(12453) + P(12435)$$

Examples

Example 3: Non-trivial diagram



$$= \frac{(\langle 15\rangle\langle 26\rangle\langle 34\rangle - \langle 14\rangle\langle 25\rangle\langle 36\rangle)^2}{\langle 12\rangle\langle 13\rangle\langle 14\rangle\langle 15\rangle\langle 23\rangle\langle 25\rangle\langle 26\rangle\langle 34\rangle\langle 36\rangle\langle 45\rangle\langle 46\rangle\langle 56\rangle}$$

$$= P(126435) + P(123564) + P(123456) + P(125463) + P(126453) + P(125364)$$

Note that the complete expression is very compact!

Non-planar Yang-Mills amplitudes

[Arkani-Hamed, Bourjaily, Cachazo, Postnikov, JT, to appear]

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However, at this moment we do not know how to expand the non-planar $\mathcal{N} = 4$ SYM amplitude in terms of on-shell diagrams.

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2. Make this property manifest in a new expansion.
3. Find a formulation which makes all symmetries manifest.

Three step strategy

In the case of planar $\mathcal{N} = 4$ SYM:

1. New property found in the standard formulation: dual conformal symmetry later unified to Yangian symmetry.
2. New expansion which makes this property manifest term-by-term: on-shell diagrams and Positive Grassmannian.
3. Complete reformulation which makes all properties manifest: Amplituhedron.

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Motivation from planar sector:

- Yangian symmetry? Not directly as this requires cyclic symmetry, perhaps some modification but hard to test now.
- Logarithmic singularities: this looks very reasonable!

Conjecture

This is our conjecture:

The complete $\mathcal{N} = 4$ SYM amplitudes have only logarithmic singularities and no poles at infinity.

There is a difficulty with testing this conjecture on an arbitrary representation of the amplitude: absence of the integrand – we can not combine pieces together.

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Temporary strategy: stay with the local expansion, use the basis which makes these two properties manifest term-by-term and prove that we can write the amplitude in this basis.

We will do it up to 3-loops at 4pt.

Logarithmic singularities

The form has only logarithmic singularities if near any pole $x_i \to a$,

$$\Omega(x_1, \ldots, x_m) \to \frac{dx_i}{x_i - a}\, \Omega(x_1 \ldots \hat{x}_i \ldots x_m)$$

We can change variables $x_i \to f_i^{(k)}(x_j)$,

$$\Omega = \sum_k \mathrm{dlog}\, f_1^{(k)}\ \mathrm{dlog}\, f_2^{(k)}\ \ldots \mathrm{dlog}\, f_m^{(k)}$$

where we denote $\mathrm{dlog}\, x \equiv dx/x$. Example of such a form is $\Omega(x) = dx/x \equiv \mathrm{dlog}\, x$, while $\Omega(x) = dx$ or $\Omega(x) = dx/x^2$ are not.

Example of 2-form:

$$\Omega(x, y) = \frac{dx\, dy}{xy(x + y + 1)} = \mathrm{dlog}\left(\frac{x}{x + y + 1}\right)\ \mathrm{dlog}\left(\frac{y}{x + y + 1}\right)$$

but not $\Omega(x, y) = dx\, dy/xy(x + y)$ as near $x = 0$: dy/y^2.

Poles at infinity

Logarithmic forms for loop integrals: take residues and study if positions of loop momentum $\ell \to \infty$. One-loop examples:

$$I_2 = \frac{d^4\ell}{\ell^2(\ell - k_1 - k_2)^2}, \qquad I_3 = \frac{d^4\ell\, s}{\ell^2(\ell - k_1)^2(\ell - k_1 - k_2)^2}$$

$$I_4 = \frac{d^4\ell\, st}{\ell^2(\ell - k_1)^2(\ell - k_1 - k_2)^2(\ell + k_4)^2}$$

Parametrize the loop momentum:

$$\ell = \alpha_1\lambda_1\widetilde{\lambda}_1 + \alpha_2\lambda_2\widetilde{\lambda}_2 + \alpha_3\lambda_1\widetilde{\lambda}_2 + \alpha_4\lambda_2\widetilde{\lambda}_1$$

and study I_2, I_3, I_4 as functions of α_i. The result is:

- Bubble integral does not have logarithmic singularities.
- Triangle has log singularities with a pole for $\alpha_3 \to \infty$.
- Only the box integral has both properties.

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Examples of UV divergent integrals:

$$\int \frac{d^4\ell}{\ell^2(\ell+p_1+p_2)^2}, \quad \int \frac{d^4\ell}{(\ell\cdot p_1)(\ell\cdot p_2)(\ell\cdot p_3)(\ell\cdot p_4)}$$

Examples of UV finite integrals:

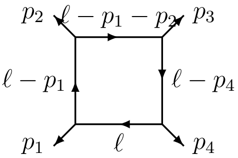
$$\int \frac{d^4\ell}{\ell^2(\ell+p_1)^2(\ell+p_1+p_2)^2}, \quad \int \frac{d^4\ell}{\ell^2(\ell+p_1)^2(\ell+p_1+p_2)^2(\ell-p_4)^2}$$

One-loop amplitude

In the local expansion we get sum over permutations over $\mathcal{I}$

$$\mathcal{A}_4^{1-loop} = \left(\frac{[34][41]}{\langle 12\rangle\langle 23\rangle}\right) \cdot \sum_\sigma C_\sigma \, \mathcal{I}_\sigma$$

where $\mathcal{I}$ is a 0-mass box integral,



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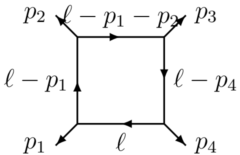

This integral has logarithmic singularities and no poles at infinity.

$$\mathrm{dlog}\,\frac{\ell^2}{(\ell-\ell^*)^2}\;\mathrm{dlog}\,\frac{(\ell-p_1)^2}{(\ell-\ell^*)^2}\;\mathrm{dlog}\,\frac{(\ell-p_1-p_2)^2}{(\ell-\ell^*)^2}\;\mathrm{dlog}\,\frac{(\ell+p_4)^2}{(\ell-\ell^*)^2}$$

and the one-loop amplitude preserves this property.

Two-loop amplitude

In the two-loop case the amplitude is written using two integrals.

$$\mathcal{A}_4^{2-loop} = \left(\frac{[34][41]}{\langle 12\rangle\langle 23\rangle}\right) \cdot \sum_{\sigma} \left[C_\sigma^{(P)}\,\mathcal{I}_\sigma^{(P)} + C_\sigma^{(NP)}\,\mathcal{I}_\sigma^{(NP)}\right]$$

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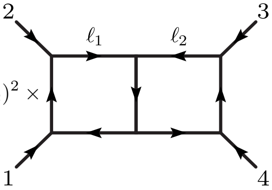
The planar double box $\mathcal{I}^{(P)}$

$$\mathcal{I}^{(P)}_{1,2,3,4} \equiv (p_1 + p_2)^2 \times$$



,

can be directly written in the dlog form

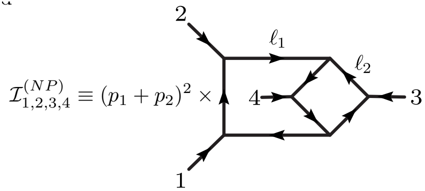
$$\mathrm{dlog}\, \alpha_1 \;\, \mathrm{dlog}\, \alpha_2 \;\, \mathrm{dlog}\, \alpha_3 \ldots \mathrm{dlog}\, \alpha_8$$

where

$$\begin{aligned}
\alpha_1 &\equiv \ell_1^2/(\ell_1 - \ell_1^*)^2, & \alpha_5 &\equiv \ell_2^2/(\ell_2 - \ell_2^*)^2, \\
\alpha_2 &\equiv (\ell_1 - p_2)^2/(\ell_1 - \ell_1^*)^2, & \alpha_6 &\equiv (\ell_1 + \ell_2)^2/(\ell_2 - \ell_2^*)^2, \\
\alpha_3 &\equiv (\ell_1 - p_1 - p_2)^2/(\ell_1 - \ell_1^*)^2, & \alpha_7 &\equiv (\ell_2 - p_3)^2/(\ell_2 - \ell_2^*)^2, \\
\alpha_4 &\equiv (\ell_1 + p_3)^2/(\ell_1 - \ell_1^*)^2, & \alpha_8 &\equiv (\ell_2 - p_3 - p_4)^2/(\ell_2 - \ell_2^*)^2,
\end{aligned}$$

Non-planar double box

The non-planar double box $\mathcal{I}_\sigma^{(NP)}$



does not have logarithmic singularities. For example, do quadruple cut on ℓ_2 and triple cut on $\ell_1 = xp_2$ we get

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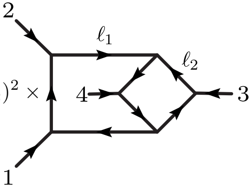

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Proposal: there are cancelations between terms and the amplitude is indeed logarithmic.

We want to keep the same diagram and just change its numerator.

Non-planar double box

And indeed such a numerator exists!

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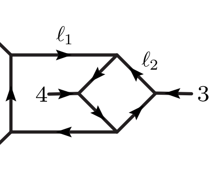

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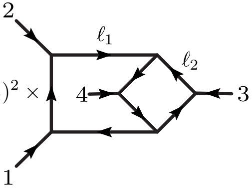


We change $(p_1 + p_2)^2 \to (\ell_1 + p_3)^2 + (\ell_1 + p_4)^2$.

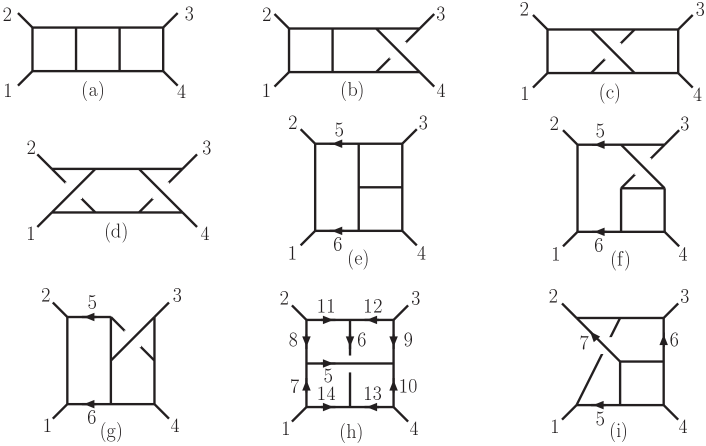
The difference cancels in the color sum and all terms in the expansion have logarithmic singularities and no poles at infinity.

There is also a dlog form which contains several terms because leading singularities of this integral are not unit.

Three-loop amplitude

The three-loop amplitude is a sum over permutations of nine master integrals with proper color factors,

[Bern, Carrasco, Dixon, Johansson, Kosower, Roiban, 2007]



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We can try to repeat the exercise and find the numerators for these integrals which give integrals with logarithmic singularities and no poles at infinity.

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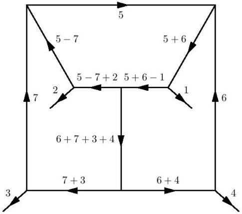
But it is indeed possible to do it!

Three-loop amplitude

As an example, I will show the most annoying diagram which has three pentagons in it:



The original numerator is only linear in loop momenta,

$$N^{(h)} = s_{12}(\ell_6 + \ell_7 + p_3 + p_4) \cdot (p_1 + p_2) + s_{23}(\ell_5) \cdot (p_2 + p_3) + s_{12}s_{23}$$

but the integral with this numerator has double poles.

Three-loop amplitude

We impose the complete set of constraints and get two independent numerators:

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N_1^{(h)} &= (\ell_5 + p_2 + p_3)^2 (\ell_6 + \ell_7)^2 - \ell_5^2 (\ell_6 + \ell_7 - p_1 - p_2)^2 \\
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Try to expand the amplitude in this basis . . . and succeed!

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This is a very strong evidence for the conjecture that this property holds to all loop orders.

Now we would like to proceed to Step 2 of our process: make these properties manifest in some new expansion to all loop orders.

The on-shell diagrams are the natural candidates as they manifestly have both properties.

Supergravity

[Arkani-Hamed, Bourjaily, Cachazo, JT, to appear]

[Bern, Herrmann, Litsey, Stankowicz, JT, in progress]

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- They are well-defined, we can obtain them by gluing 3pt vertices.
- We can associate a cell in $G(k, n)$ for each reduced diagram.
- However, we do not know what is the form to be associated with a diagram – it is not a logarithmic form.

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At 3-loops we better use BCJ representation of the gravity amplitude:

$$A_{GR} = \sum \frac{N_{YM}^{(new)} \cdot N_{YM}^{(BCJ)}}{D}$$

where the BCJ numerator $N_{YM}^{(BCJ)}$ is only linear in loop momenta.

Supergravity

Based on this expansion we can easily prove that the gravity amplitude still has logarithmic singularities while the manifest absence of poles at infinity is lost.

For $\ell = \alpha_1 p_1 + \alpha_2 p_2 + \alpha_3 p_3 + \alpha_4 p_4$ we get

$$\frac{N_{YM}^{(new)}}{D} = \frac{N_p(\alpha)}{N_q(\alpha)} \sim \frac{1}{\alpha(\alpha+1)} \qquad q \geq p+2$$

The BCJ numerator is $N_{YM}^{(BCJ)} \sim \alpha$ and therefore

$$A_{GR} \sim \frac{1}{\alpha}$$

Pole at infinity: $\alpha \to \infty$. It still might cancel between terms but the preliminary checks show they do not.

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Logarithmic singularities and no poles at infinity guaranteed UV finiteness in the case of $\mathcal{N} = 4$ SYM.

If the poles at infinity do not cancel for $\mathcal{N} = 8$ SUGRA, the theory is either UV divergent or there is some other mechanism which implies UV finiteness.

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Similar study for $\mathcal{N} = 8$ Supergravity: logarithmic singularities up to 3-loops. Further studies at higher loops should also prove/disprove the finiteness conjecture.

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..... and happy birthday, Andrew!