The Scattering Equations, their Properties and Proofs

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(work with Peter Goddard, IAS Princeton)

1402.7374 [hep-th], *The Polynomial Form of the Scattering Equations*

1311.5200 [hep-th], *Proof of the Formula of Cachazo, He and Yuan for Yang-Mills Tree Amplitudes in Arbitrary Dimension*

Freddy Cachazo, Song He, and Ellis Yuan (CHY)

1309.0885 [hep-th],
Scattering of Massless Particles: Scalars, Gluons and Gravitons

1307.2199 [hep-th],
Scattering of Massless Particles in Arbitrary Dimensions

1306.6575 [hep-th],
Scattering Equations and KLT Orthogonality

Outline

- Tree amplitudes from the Scattering Equations in any dimension
- Möbius invariance and massive Scattering Equations
- Proof of the equivalence with ϕ^3 and Yang-Mills field theories
- Polynomial form of the Scattering Equations
- Polynomial form of the Scattering Equations at One Loop

Tree Amplitudes

$$\mathcal{A}(k_1, k_2, \ldots, k_N) = \oint_{\mathcal{O}} \Psi_N(z, k, \epsilon) {\prod_{a\in A}}' \frac{1}{f_a(z,k)} \prod_{a\in A} \frac{dz_a}{(z_a - z_{a+1})^2} / d\omega,$$

$\mathcal{O}$ encircles the zeros of $f_a(z, k)$,

$$f_a(z,k) \equiv \sum_{\substack{b\in A \\ b\neq a}} \frac{k_a \cdot k_b}{z_a - z_b} = 0 \qquad \text{The Scattering Equations}$$

(*Cachazo*, *He*, *Yuan* 2013) . . . (*Fairlie*, *Roberts* 1972)

$$k_a^2 = 0, \quad \sum_{a\in A} k_a^\mu = 0, \quad A = \{1, 2, \ldots N.\}$$

DG proved $\mathcal{A}(k_1, k_2, \ldots k_n)$ are ϕ^3 and Yang-Mills gluon field theory tree amplitudes , as conjectured by CHY.

Möbius Invariance

$$z_a \to \frac{\alpha z_a+\beta}{\gamma z_a+\delta},$$

$$\mathcal{A}(k_1, k_2, \ldots, k_N) = \oint_{\mathcal{O}} \Psi_N(z,k,\epsilon) {\prod_{a\in A}}' \frac{1}{f_a(z,k)} \prod_{a\in A} \frac{dz_a}{(z_a - z_{a+1})^2} / d\omega$$

$${\prod_{a\in A}}' \frac{1}{f_a(z,k)} \equiv (z_1 - z_2)(z_2 - z_N)(z_N - z_1) \prod_{\substack{a\in A \\ a\neq 1,2,N}} \frac{1}{f_a(z,k)}$$

$$\to \prod_{a\in A} \frac{(\alpha\delta - \beta\delta)}{(\gamma z_a + \delta)^2} {\prod_{a\in A}}' \frac{1}{f_a(z,k)},$$

$\Psi_N(z,k,\epsilon)$ is Möbius invariant,
$\Psi_N = 1$ for ϕ^3, $\quad \Psi_N = \prod_{a\in A}(z_a - z_{a+1}) \times$ Pfaffian for Yang-Mills

The integrand and the Scattering Equations are Möbius invariant (CHY).

Massive Scattering Equations $\quad \widehat{f}_a(z,k)=0, \quad k_a^2=m^2$

$$U(z,k)\equiv\prod_{a<b}(z_a-z_b)^{-k_a\cdot k_b}\prod_{a\in A}(z_a-z_{a+1})^{-\frac{m^2}{2}}$$ is Möbius invariant,

$$\frac{\partial U}{\partial z_a}=-\widehat{f}_a U,\quad \widehat{f}_a(z,k)=\sum_{\substack{b\in A\\ b\neq a}}\frac{k_a\cdot k_b}{z_a-z_b}+\frac{m^2}{2(z_a-z_{a+1})}+\frac{m^2}{2(z_a-z_{a-1})},$$

implying $\widehat{f}_a(z)\to\widehat{f}_a(z)\dfrac{(\gamma z_a+\delta)^2}{(\alpha\delta-\beta\gamma)}$.

The infinitesimal transformations $\delta z_a=\epsilon_1+\epsilon_2 z_a+\epsilon_3 z_a^2$,

$U(z+\delta z)\sim U(z)+\dfrac{\partial U}{\partial z_a}\delta z_a$, so the $\widehat{f}_a$ satisfy the three relations

$$\sum_{a\in A}\widehat{f}_a=0,\quad \sum_{a\in A}z_a\widehat{f}_a=0,\quad \sum_{a\in A}z_a^2\widehat{f}_a=0.$$

There are $N-3$ independent Scattering Equations $\widehat{f}_a=0$.

Fixing $z_1=\infty, z_2=1, z_N=0$, there are $N-3$ variables,

and generally $(N-3)!$ solutions $z_a(k)$. $\quad \hat{f}=f$ when $m^2=0$.

Total Amplitudes

For example, $N = 4$,

$$A^{abcd}(k_1, k_2, k_3, k_4) = g^2\Big(f_{abe}f_{ecd}\,\frac{n_s}{s} + f_{bce}f_{ead}\,\frac{n_t}{t} + f_{cae}f_{ebd}\,\frac{n_u}{u}\Big)$$

$$= g^2\Big(\big(tr(T_aT_bT_cT_d) + tr(T_dT_cT_bT_a)\big)\,A(1234)$$

$$+ \big(tr(T_aT_cT_dT_b) + tr(T_bT_dT_cT_a)\big)\,A(1342)$$

$$+ \big(tr(T_aT_dT_bT_c) + tr(T_cT_bT_dT_a)\big)\,A(1423)\Big),$$

$$n_s = \big(\epsilon_1\cdot\epsilon_2(k_1{-}k_2)_\alpha + 2\epsilon_1\cdot k_2\epsilon_{2\alpha} - 2\epsilon_2\cdot k_1\epsilon_{1\alpha}\big)$$

$$\times\big(\epsilon_3\cdot\epsilon_4(k_3-k_4)^\alpha + 2\epsilon_3\cdot k_4\epsilon_4^\alpha - 2\epsilon_4\cdot k_3\epsilon_3^\alpha\big)$$

$$+ \big(\epsilon_1\cdot\epsilon_3\epsilon_2{\cdot}\epsilon_4 - \epsilon_1\cdot\epsilon_4\epsilon_2\cdot\epsilon_3\big)\,s,$$

$$A(1234) = \frac{n_s}{s} + \frac{n_t}{t}. \qquad s = (k_1+k_2)^2, t = (k_2+k_3)^2, u = (k_1+k_3)^2$$

$$\mathcal{A}(k_1, k_2, k_3, k_4) = A(1234).$$

A Single Scalar Field, Massless ϕ^3

A single massless scalar field, $\Psi_N = 1$.

$$\mathcal{A}^{\phi}(k_1, k_2, \ldots, k_N) = \oint_{\mathcal{O}} \prod_{a\in A}{}' \frac{1}{f_a(z,k)} \prod_{a\in A} \frac{dz_a}{(z_a - z_{a+1})^2} / d\omega$$

$$\mathcal{A}^{\phi}(k_1, k_2, k_3, k_4) = \frac{1}{s} + \frac{1}{t},$$

$$A^{\text{total}} = \mathcal{A}^{\phi}(k_1, k_2, k_3, k_4) + \mathcal{A}^{\phi}(k_1, k_3, k_2, k_4) + \mathcal{A}^{\phi}(k_1, k_4, k_2, k_3)$$

$$= 2\left(\frac{1}{s} + \frac{1}{t} + \frac{1}{u}\right)$$

Proof of the Formula of CHY for Massless ϕ^3

$$A_N^{\phi}(\zeta) = A_N^{\phi}(k_1, k_2 + \zeta\ell, k_3, \ldots, k_{N-1}, k_N - \zeta\ell),$$

For $\ell^2 = \ell \cdot k_2 = \ell \cdot k_N = 0$, these shifted, ordered field theory tree amplitudes have simple poles in ζ, and $A_N^{\phi}(\zeta) \to 0$ as $\zeta \to \infty$.

$$A_N^{\phi}(\zeta) = -\sum_i \frac{\mathrm{Res}_{\zeta_i} A_N^{\phi}}{\zeta_i - \zeta}$$

The poles ζ_i occur where $(\pi_m^{\zeta})^2 = 0$ or $(\bar{\pi}_m^{\zeta})^2 = 0$, *i.e.* at

$$\zeta = s_m/2\pi_m \cdot \ell \equiv \zeta_m^L, \text{ and } \zeta = -\bar{s}_m/2\bar{\pi}_m \cdot \ell \equiv \zeta_m^R, \quad 3 \leq m \leq N-1,$$

with residues given by

$$\mathrm{Res}_{\zeta_m^R} A_N^{\phi} = A_m^{\phi}(k_1, k_2^{\zeta_m^R}, k_3, \ldots, k_{m-1}, -\bar{\pi}_m^{\zeta_m^R}) \frac{1}{2\bar{\pi}_m \cdot \ell}$$
$$\times A_{N-m+2}^{\phi}(\bar{\pi}_m^{\zeta_m^R}, k_m, \ldots, k_{N-1}, k_N^{\zeta_m^R}),$$

$\pi_m \equiv -k_2 - k_3 - \ldots - k_m, \quad \bar{\pi}_m \equiv -k_m - k_3 - \ldots - k_N;\ s_m = \pi_m^2,\ \bar{s}_m = \bar{\pi}_m^2.$

$$A^\phi(k_1,k_2,\ldots,k_N) = A_N^\phi(\zeta=0)$$
$$= -2\sum_{m=3}^{N-1}\left[\frac{2\pi_m\cdot\ell}{s_m}\mathrm{Res}_{\zeta_m^L}A_N^\phi - \frac{2\bar{\pi}_m\cdot\ell}{\bar{s}_m}\mathrm{Res}_{\zeta_m^R}A_N^\phi\right] \qquad *$$

which determines $A^\phi(k_1,\ldots k_N)$ for $N>3$ from $A^\phi(k_1,k_2,k_3)=1$.

Our proof is to show $A^\phi = \mathcal{A}^\phi$ satisfies $*$.

$$\mathcal{A}_N^\phi(\zeta) \sim \oint \frac{\prod_{a=3}^{N-2} z_a \prod_{a=4}^{N-1}(1-z_a)}{(1-z_3)z_{N-1}} \prod_{b=5}^{N-1}\prod_{a=3}^{b-2}(z_a-z_b)^2 \prod_{a=3}^{N-1}\frac{dz_a}{f_a(z,\zeta)}$$

A pole at ζ_m^R comes from the integration region $z_a \to 0$, $m \le a \le N-1$. Let $z_a = x_a z_m$, $z_m \to 0$,

$$\prod_{a=3}^{N-1} dz_a = \prod_{a=3}^{m-1} dz_a \; dz_m \; \prod_{a=m+1}^{N-1} dx_a,$$

$$\begin{aligned}\mathrm{Res}_{\zeta_m^R}\mathcal{A}_N^{\phi} = \mathcal{A}_m^{\phi}(k_1, k_2^{\zeta_m^R}, k_3, \ldots, k_{m-1}, -\bar{\pi}_m^{\zeta_m^R})\frac{1}{2\bar{\pi}_m \cdot \ell} \\ \times \mathcal{A}_{N-m+2}^{\phi}(\bar{\pi}_m^{\zeta_m^R}, k_m, \ldots, k_{N-1}, k_N^{\zeta_m^R}),\end{aligned}$$

Similarly for $\mathrm{Res}_{\zeta_m^L}\mathcal{A}_N^{\phi}$.

So proving the formula for $\mathcal{A}^{\phi}(k_1, \ldots, k_N)$ by induction.

Proof for Pure Gauge Theory

$$\mathcal{A}_N^{\mathrm{YM}}(\zeta) \sim \oint \Psi_N^o \frac{\prod_{a=3}^{N-2} z_a \prod_{a=4}^{N-1}(1-z_a)}{(1-z_3)z_{N-1}} \prod_{b=5}^{N-1} \prod_{a=3}^{b-2} (z_a - z_b)^2 \prod_{a=3}^{N-1} \frac{dz_a}{f_a(z,\zeta)}$$

where the only difference from the scalar case is Ψ_N^o, which is related to the Pfaffian of the antisymmetric matrix M_N with the 2nd and Nth rows and columns removed,

$$\Psi_N^o = (-1)^N \mathrm{Pf}\, \mathrm{M}_N(z; k^\zeta; \epsilon^\zeta)_{(2,N)} \prod_{a=1}^{N} (z_a - z_{a+1}),$$

$$\det \mathrm{M} \equiv (\mathrm{Pf}\ \ \mathrm{M})^2,$$

$$\epsilon_2^{\zeta+} = \bar{\ell} - 2(\zeta / k_2 \cdot k_N) k_N, \ \ \epsilon_2^{\zeta-} = \ell; \ \ \epsilon_4^{\zeta\pm},$$

$$\bar{\ell}^2 = \bar{\ell} \cdot k_2 = \bar{\ell} \cdot k_N = 0, \ell \cdot \bar{\ell} = 2.$$

All singularities in Ψ_N^o are canceled by the numerator. Ψ_N^o factorizes at the poles in the integrand, $\zeta_m^{L,R}$, since the Pfaffian does. As $z_m \to 0$,

$$\begin{aligned}
&\text{Pf } \mathrm{M}_N(k_1, \ldots, k_N; \epsilon_1, \ldots, \epsilon_N; z_3, \ldots, z_{N-1})_{(2,N)} \\
&\sim \sum_s \text{Pf, } \mathrm{M}_m(k_1, \ldots, k_{m-1}, -\bar{\pi}_m; \epsilon_1, \ldots, \epsilon_{m-1}, \epsilon^s; z_3, \ldots, z_{m-1})_{(2,m)} \\
&\times \text{Pf } \mathrm{M}_{N-m+2}(\bar{\pi}_m, k_m, \ldots, k_N; \epsilon^s, \epsilon_m, \ldots, \epsilon_N; x_{m+1}, \ldots, x_{N-1})_{(1,N-m+2)},
\end{aligned}$$

and

$$\prod_{a=2}^{N-1}(z_a - z_{a+1}) \to z_{m-1} z_m^{N-m} \prod_{a=2}^{m-2}(z_a - z_{a+1}) \prod_{a=m}^{N-1}(x_a - x_{a+1})$$

This demonstrates that $\mathcal{A}_N^{\text{YM}}(\zeta = 0)$ satisfies the BCFW recurrence relation, so that $\mathcal{A}^{\text{YM}}(k_1, \ldots k_N)$, computed from the scattering equations, are equal to the Yang Mills field theory tree amplitudes.

Equivalence between
Twistor String Equations and the Scattering Equations

4-dimensional momenta $k_{a\alpha\dot\alpha} = \pi_{a\alpha}\bar\pi_{a\dot\alpha}, \quad 1 \le a \le N;\ \alpha, \dot\alpha = 1, 2.$
$\{a \in A : a = i \in \mathcal{P},\ r \in \mathcal{N}, \quad m + n = N\}, \quad \rho_a \equiv z_a.$

Link variables $c_{ir} \equiv \frac{\lambda_i}{\lambda_r(z_i - z_r)}$ satisfy:

$$\pi_i^\alpha = \sum\nolimits_{r\in\mathcal{N}} c_{ir}\pi_r^\alpha, \quad -\bar\pi_{r\dot\alpha} = \sum\nolimits_{i\in\mathcal{P}} \bar\pi_{i\dot\alpha} c_{ir}.$$

$$2\sum_b \frac{k_i \cdot k_b}{z_i - z_b} = \sum_r \frac{\langle\pi_i, \pi_r\rangle[\bar\pi_i, \bar\pi_r]}{z_i - z_r} + \sum_j \frac{\langle\pi_i, \pi_j\rangle[\bar\pi_i, \bar\pi_j]}{z_i - z_j} = 0$$

Similarly $\sum_b \frac{k_r \cdot k_b}{z_r - z_b} = 0.$

$2k_a \cdot k_b = \langle\pi_a, \pi_b\rangle[\bar\pi_a, \bar\pi_b]; \quad \langle\pi_a, \pi_b\rangle \equiv \pi_{a\alpha}\pi_b^\alpha, \ \ [\bar\pi_a, \bar\pi_b] \equiv \bar\pi_{a\dot\alpha}\pi_b^{\dot\alpha}$

Polynomial Form for the Scattering Equations

For a subset $U \subset A$,

$$k_U \equiv \sum_{a \in U} k_a, \qquad z_U \equiv \prod_{b \in U} z_b,$$

then the Scattering Equations

$$\sum_{\substack{b \in A \\ b \neq a}} \frac{k_a \cdot k_b}{z_a - z_b} = 0$$

are equivalent to the homogeneous polynomial equations

$$\sum_{\substack{U \subset A \\ |U| = m}} k_U^2 z_U = 0, \quad 2 \leq m \leq N - 2,$$

where the sum is over all $\frac{N!}{m!(N-m)!}$ subsets $U \subset A$ with m elements.

Proof of the Polynomial Form of the Scattering Equations

$$p^\mu(z) \equiv \sum_{a\in A} \frac{k_a^\mu}{z - z_a}, \quad \sum_a k_a^\mu = 0, \quad k_a^2 = 0,$$

$$p^2(z) = \sum_{a,b} \frac{k_a \cdot k_b}{(z - z_a)(z - z_b)} = \frac{1}{2} \sum_a \frac{1}{z - z_a} \sum_{b\neq a} \frac{k_a \cdot k_b}{(z_a - z_b)} = 0$$

$$\begin{aligned} 2p^2(z) \prod_{c\in A} (z - z_c) &= \sum_{a,b\in A} 2k_a \cdot k_b \prod_{\substack{c\in A \\ c\neq a,b}} (z - z_c) \\ &= \sum_{m=0}^{N-2} z^{N-m-2} \sum_{\substack{U\subset A \\ |U|=m}} z_U \sum_{\substack{S\subset \overline{U} \\ |S|=2}} k_S^2 = 0 \end{aligned}$$

where $\overline{U} = \{b \in A : b \notin U\}$. Using $\sum_{\substack{S\subset \overline{U} \\ |S|=2}} k_S^2 = k_{\overline{U}}^2 = k_U^2$, then

$\widetilde{h}_m \equiv \sum_{\substack{U\subset A \\ |U|=m}} k_U^2 z_U = 0.$

The Scattering Equations are the Unique Polynomial Equations that are Möbius Invariant

L_{-1} denotes the generator of translations,

$$L_{-1} = -\sum_{a\in A} \frac{\partial}{\partial z_a}, \qquad L_{-1}\,\tilde{h}_m = -(N-m-1)\tilde{h}_{m-1}\,,$$

L_1, special conformal transformations

$$L_1 = -\sum_{a\in A} z_a^2 \frac{\partial}{\partial z_a} + \Sigma_1^A, \qquad L_1\,\tilde{h}_m = (m-1)\tilde{h}_{m+1},$$

L_0, scale transformations

$$L_0 = -\sum_{a\in A} z_a \frac{\partial}{\partial z_a} + \frac{N}{2}, \qquad L_0\,\tilde{h}_m = (\tfrac{1}{2}N-m)\tilde{h}_m\,,$$

so that $\quad [L_1, L_{-1}] = 2L_0, \qquad [L_0, L_{\pm1}] = \mp L_{\pm1}.$

The $\tilde{h}_m$, $2 \le m \le N-2$, form an $(N-3)$-dimensional multiplet of the Möbius algebra, *i.e.* a representation of 'Möbius spin' $\frac{1}{2}N-2$. The equations $\tilde{h}_m(z_1,\ldots,z_n) = \sum_{\substack{U\subset A\\ |U|=m}} k_U^2 z_U = 0$ determine a discrete set of points (up to Möbius invariance).

$z_1 \to \infty, z_2$ fixed, $z_N \to 0,$

Amplitudes in terms of Polynomial Constraints

$$\mathcal{A}_N = \oint_{\mathcal{O}} \Psi_N(z,k) \frac{z_2}{z_{N-1}} \prod_{m=1}^{N-3} \frac{1}{h_m(z,k)} \prod_{2 \le a < b \le N-1} (z_a - z_b) \prod_{a=2}^{N-2} \frac{z_a dz_{a+1}}{(z_a - z_{a+1})^2}.$$

$$h_m = \lim_{z_1 \to \infty} \frac{\widetilde{h}_{m+1}}{z_1} = \frac{1}{m!} \sum_{\substack{a_1, a_2, \ldots, a_m \neq 1, N \\ a_i \text{ uneq.}}} k^2_{1a_1 \ldots a_m} z_{a_1} z_{a_2} \ldots z_{a_m}, \qquad 1 \le m \le N-3,$$

The $N-3$ polynomial equations $h_m = 0$, of degree m,
linear in each z_a individually,
are equivalent to the Scattering Equations.

By Bézout's theorem, they determine (N-3)! solutions for the ratios of the $z_2, \ldots, z_{N-1}$.

Solutions to the Scattering Equations

$N = 4$

$$h_1 = k_{12}^2 z_2 + k_{13}^2 z_3 = 0, \qquad z_3/z_2 = -k_{12}^2/k_{13}^2 = -k_1 \cdot k_2 / k_1 \cdot k_3.$$

$N = 5$

$$h_1 = k_{12}^2 z_2 + k_{13}^2 z_3 + k_{14}^2 z_4 = 0,$$
$$h_2 = k_{123}^2 z_2 z_3 + k_{124}^2 z_2 z_4 + k_{134}^2 z_3 z_4 = 0,$$

eliminating z_4 yields a quadratic equation for z_3/z_2.
This can be written as

$$\begin{vmatrix} h_1 & h_2 \\ \frac{\partial h_1}{\partial z_2} & \frac{\partial h_2}{\partial z_2} \end{vmatrix} = 0.$$

$N = 6$ write $(x, y, z, u) = (z_2, z_3, z_4, z_5)$

$$h_1 = k_{12}^2 x + k_{13}^2 y + k_{14}^2 z + k_{15}^2 = 0,$$
$$h_2 = k_{123}^2 xy + k_{124}^2 xz + k_{134}^2 yz + k_{125}^2 xu + k_{135}^2 yu + k_{145}^2 zu = 0,$$
$$h_3 = k_{1234}^2 xyz + k_{1235}^2 xyu + k_{1245}^2 xzu + k_{1345}^2 yzu = 0,$$

eliminating x, y yields a sextic equation for z/u.
This can be written

$$\begin{vmatrix} h_1 & h_2 & h_3 & 0 & 0 & 0 \\ h_1^x & h_2^x & h_3^x & 0 & 0 & 0 \\ h_1^y & h_2^y & h_3^y & h_1 & h_2 & h_3 \\ h_1^{xy} & h_2^{xy} & h_3^{xy} & h_1^x & h_2^x & h_3^x \\ 0 & 0 & 0 & h_1^y & h_2^y & h_3^y \\ 0 & 0 & 0 & h_1^{xy} & h_2^{xy} & h_3^{xy} \end{vmatrix} = 0,$$

$$h_m^x = \frac{\partial h_m}{\partial x}, \qquad h_m^{xy} = \frac{\partial^2 h_m}{\partial x \partial y}, \qquad \text{etc.}$$

The one-loop Scattering Equations

$$P^\mu(\nu,\tau) \equiv p^\mu + \sum_{a\in A} k_a^\mu \zeta(\nu-\nu_a) = k^\mu + \frac{1}{2}\sum_{a\in A} k_a^\mu \frac{\wp'(\nu)+\wp'(\nu_a)}{\wp(\nu)-\wp(\nu_a)},$$

where $p^\mu = k^\mu + \sum_{a\in A} k_a^\mu \zeta(\nu_a)$.

$P^\mu(\nu,\tau)$ is defined on the torus,
$P^\mu(\nu+1,\tau) = P^\mu(\nu+\tau,\tau) = P^\mu(\nu,\tau)$, when $\sum_{a\in A} k_a^\mu = 0$.
$P(\nu,\tau)^2$ has no poles when $k_a^2 = 0$ and

$$f_a = p\cdot k_a + \sum_{b\in A, b\neq a} k_a\cdot k_b\,\zeta(\nu_a-\nu_b) = 0, \qquad a\in A.$$

Then $P(\nu,\tau)^2 = k^2$ is a constant. For modular invariance, $k^2=0$.

$$\sum_{a\in A} f_a = 0.$$

The one-loop Scattering Equations are the N equations:

$$f_a = 0, \qquad k^2 = 0.$$

see also Adamo, Casali and Skinner 1312.3828 [hep-th],

Gross and Mende, Nucl. Phys. B ***303****, 407 (1988).*

Review of Elliptic Functions: Functions on the Torus

The Weierstrass $\wp(\nu,\tau)$ function,

$$\wp(\nu+1)=\wp(\nu), \quad \wp(\nu+\tau)=\wp(\nu),$$

$$\wp(\nu)=-\zeta'(\nu), \quad \zeta(\nu)=\frac{\theta_1'(\nu,\tau)}{\theta_1(\nu,\tau)}+2\eta(\tau)\nu, \quad \eta(\tau)=-\frac{\theta_1'''(0,\tau)}{6\,\theta_1'(0,\tau)},$$

$$\zeta(\nu+1)=\zeta(\nu)+2\eta(\tau); \qquad \zeta(\nu+\tau)=\zeta(\nu)-2\pi i+2\eta(\tau)\tau.$$

Modular Properties:

$$\zeta(\nu,\tau+1)=\zeta(\nu,\tau), \qquad \zeta(\frac{\nu}{\tau},-\frac{1}{\tau})=\tau\,\zeta(\nu,\tau),$$

$$\wp(\nu,\tau+1)=\wp(\nu,\tau), \qquad \wp(\frac{\nu}{\tau},-\frac{1}{\tau})=\tau^2\,\wp(\nu,\tau),$$

$$\wp'(\nu,\tau+1)=\wp'(\nu,\tau), \qquad \wp'(\frac{\nu}{\tau},-\frac{1}{\tau})=\tau^3\,\wp'(\nu,\tau).$$

The Polynomial Form of the one-loop Scattering Equations

For a subset $U \subset A$,

$$k_U \equiv \sum_{a \in U} k_a, \qquad \wp_U \equiv \prod_{b \in U} \wp_b,$$

where $\wp_b = \wp(\nu_b, \tau)$, then the one-loop Scattering Equations

$$f_a = k \cdot k_a + \frac{1}{2} \sum_{b \neq a} k_a \cdot k_b \frac{\wp'(\nu_a) + \wp'(\nu_b)}{\wp(\nu_a) - \wp(\nu_b)} = 0$$

are equivalent to the one-loop polynomial equations

$$\mathcal{A}_m = \sum_{|U|=m} k \cdot k_{\bar{U}} \wp_U + \tfrac{1}{2} \sum_{|U|=m-1} \sum_{a \in \overline{U}} k_a \cdot k_U \wp'_a \wp_U = 0, \quad 1 \leq m \leq N-1.$$

$$\mathcal{A}_1 = -\sum_a k \cdot k_a \wp_a = 0,$$

$$\mathcal{A}_2 = -\sum_{a \neq b} k \cdot k_a \wp_a \wp_b + \frac{1}{2} \sum_{a \neq b} k_a \cdot k_b \wp'_a \wp_b = 0,$$

$$\mathcal{A}_3 = -\frac{1}{2} \sum_{a,b,c \text{ unequal}} k \cdot k_a \wp_a \wp_b \wp_c + \frac{1}{2} \sum_{a,b,c \text{ unequal}} k_a \cdot k_b \wp'_a \wp_b \wp_c = 0$$

etc.

Proof of the Polynomial Form of the one-loop Scattering Equations

For $M_{ma} = \sum_{\substack{|U|=m \\ U \not\ni a}} \wp_U, \qquad 1 \le m \le N-1, \qquad M_{0a} = 1,$

$\sum_{a=1}^{N} M_{ma} f_a = \mathcal{A}_m, \qquad 0 \le m \le N-1, \qquad \mathcal{A}_0 = \sum_a k \cdot k_a,$

so that $f_a = 0$ implies $\mathcal{A}_m = 0$.

The inverse matrix is

$M^{-1}_{am} = (-1)^m \wp_a^{N-m-1} \prod_{b \neq a} (\wp_a - \wp_b)^{-1}$ implying

$$f_a = \sum_{m=0}^{N-1} M^{-1}_{am} \mathcal{A}_m,$$

so that $\mathcal{A}_m = 0$ implies $f_a = 0$.

Equivalent Polynomial Form of the one-loop Scattering Equations

$$
\begin{aligned}
&P^2(\nu,\tau)\prod_c(\wp-\wp_c)\\
&=\sum_a k\cdot k_a(\wp'+\wp'_a)\prod_{c\neq a}(\wp-\wp_c)\\
&\qquad\qquad+\frac{1}{4}\sum_{a,b}k_a\cdot k_b(\wp'+\wp'_a)(\wp'+\wp'_b)\prod_{c\neq a,b}(\wp-\wp_c)\\
&=\wp'\left[\sum_{m=1}^{N_1}\wp^{N-m-1}\mathcal{A}_m(-1)^{m+1}\right]\quad+\quad\sum_{m=0}^{N-1}\wp^{N-m-1}\mathcal{B}_m(-1)^m\\
&=0
\end{aligned}
$$

where

$$\mathcal{B}_m = \sum_{|U|=m} \sum_{a\in\overline{U}} k\cdot k_a \wp'_a \wp_U + \sum_{|U|=m+2} k_U^2 \wp_U - \frac{g_2}{4} \sum_{|U|=m} k_U^2 \wp_U + \frac{g_3}{4} \sum_{|U|=m-1} k_U^2 \wp_U$$
$$- \frac{1}{4} \sum_{|U|=m-1} \sum_{a,b\in\overline{U}} k_a \cdot k_b \wp'_a \wp'_b \wp_U,$$
$$= \sum_a M_{ma} \wp'_a f_a = 0, \qquad 0 \le m \le N-1.$$

And inverting,

$$\wp'_a f_a = \sum_{m=0}^{N-1} M^{-1}_{am} \mathcal{B}_m$$

Thus *either* (a) $\mathcal{A}_m = 0$, *or* (b) $\mathcal{B}_m = 0$, implies $f_a = 0$, $a \in A$.

Since $f_a = 0$, $a \in A$, implies both (a) and (b), it follows that (a) implies (b) and *vice versa*.

Comments

The polynomial form of the tree level Scattering Equations facilitates computation of their solutions $z_a(k)$, due to the linearity of the equations in the individual variables z_a. Bézout's theorem provides an explanation for the $(N-3)!$ solutions.

The Scattering Equations are the unique polynomial equations that are Möbius Invariant.

The Scattering Equations can be generalized to massive particles, enabling the description of tree amplitudes for massive ϕ^3 theory.

The proofs make it certain that the Scattering Equations approach is equivalent to gauge field theory at tree level.

The polynomial form of the one-loop Scattering Equations is given.