Perturbative and Non-Perturbative Properties of String Scattering Amplitudes

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New geometric structures in scattering amplitudes

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Constraints on closed string effective action from amplitude calculations

I will consider narrowly-focused aspects of the low energy effective string action obtained from closed string scattering amplitudes.

- Features of closed string perturbation theory:

  Comments on relation to supergravity field theory amplitudes.

- Non-perturbative features - duality:

  Connects perturbative with non-perturbative effects.

  Powerful constraints imposed by SUSY, Duality, Unitarity

  Connections with quantum eleven-dimensional supergravity.

- Connections with beautiful mathematics:

  Modular Forms; Automorphic forms for higher-rank groups; Multi-Zeta Values; …….

MBG, Stephen Miller, Pierre Vanhove arXiv:1404.2192
MBG, Eric D'Hoker, arXiv:1308.4597
MBG, Eric D'Hoker, Boris Pioline, Rudolfo Russo; arXiv:1405.6226

A PERSPECTIVE ON STRING SCATTERING AMPLITUDES

OPEN STRING THEORY generalises (super) Yang-Mills theory – ground state vector boson. Interpreted as excitations on a D-brane

CLOSED STRING THEORY generalises Einstein (super) gravity – ground state graviton.

Scattering Amplitudes with maximal supersymmetry are highly constrained by symmetry considerations:

In particular, DUALITIES relate theories in different regions of MODULI SPACE.

SUPERGRAVITY (low energy limit of closed string theory):

Scalar fields – (geometric and non-geometric) MODULI parameterize a symmetric space

$$G(\mathbb{R})/K(\mathbb{R})$$

groups in E_n series (real split forms)

(Cremmer, Julia)

CLOSED STRING THEORY:

Discrete identifications of scalar fields $G(\mathbb{Z})\backslash G(\mathbb{R})/K(\mathbb{R})$

DUALITY GROUP $G(\mathbb{Z})$

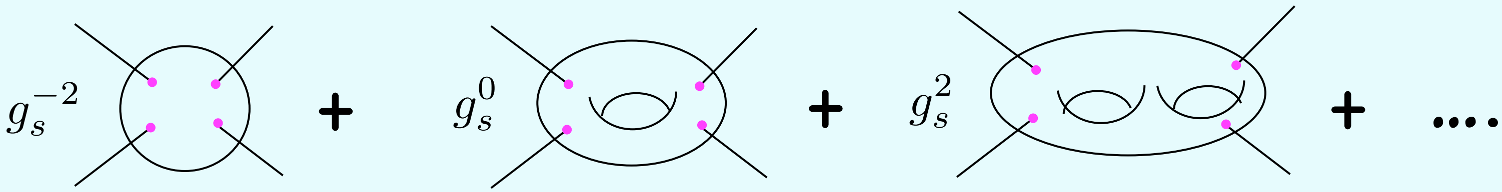
Arithmetic subgroup of $G(\mathbb{R})$ is symmetry of string theory.

RICH DEPENDENCE ON MODULI

STRING PERTURBATION THEORY: Expansion around boundary of moduli space.

e.g. in powers of $g_s = e^{\phi} \ll 1$ (c.f. FEYNMAN DIAGRAMS of quantum field theory) :

Sum of functional integrals over Riemann surfaces of arbitrary genus :



$g_s^{2h-2} \times$ (genus-h Riemann surface)

Where + ... includes nonperturbative (e.g. instanton) effects.

DUALITIES:

- Relate perturbative and non-perturbative features of amplitudes
- Dependence on moduli (or coupling constants) expressed in terms of automorphic functions for relevant duality groups.
- AdS/CFT relates closed string theory in AdS space to Yang-mills theory
  i.e. to the low energy limit of the open string theory.

THE LOW ENERGY EXPANSION OF STRING THEORY

- LOWEST ORDER TERM reproduces the results of classical supergravity

EINSTEIN-HILBERT

$\alpha' = \ell_s^2$

ℓ_s is STRING LENGTH SCALE

$$\frac{1}{\alpha'^4}\int d^{10}x\sqrt{-\det G}\, e^{-2\phi}\, R + \dots$$ several other supergravity fields

METRIC $-\ G_{\mu\nu}$

SCALAR FIELD - DILATON

$e^{-\phi} = \dfrac{1}{g_s}$ STRING COUPLING CONSTANT

Expanding the curvature in small fluctuations of the metric around D=10 Minkowski space gives contributions to "classical" MULTI-GRAVITON scattering amplitudes.

- HIGHER ORDER TERMS: $$\frac{1}{\alpha'}\int d^{10}x\,\sqrt{-\det G}\,\mathcal{F}(\phi,\dots)\,R^4 + \dots$$

MODULI-DEPENDENT COEFFICIENT

- Expansion in powers of $\alpha' R\,,\quad \alpha' D^2\,,\quad \dots$

THE LOW ENERGY EXPANSION OF (TYPE IIB) STRING THEORY

HIGHER DERIVATIVE CORRECTIONS to Einstein theory

- Four-graviton scattering contributes to higher derivative corrections of the form

$$\underbrace{R^4 \quad d^2 R^4 \quad d^4 R^4 \quad d^6 R^4}_{\text{BPS interactions}} \quad d^8 R^4$$

- N > 4 -graviton scattering contributes to:

$$R^5 \quad d^2\, R^5 \quad d^4\, R^5 \quad d^6\, R^5$$

$$R^6 \quad d^2\, R^6 \quad d^4\, R^6$$

Etc.

dilatino

- Many U(1)- violating interactions (absent in supergravity). E.g. λ^{16}

- Duality groups in $3 \leq D \leq 10$ space-time dimensions

HIGHER-RANK DUALITY GROUPS

	$SL(2,\mathbb{Z})$	$SL(2,\mathbb{Z})$	$SL(3,\mathbb{Z}) \times SL(2,\mathbb{Z})$	$SL(5,\mathbb{Z})$	$SO(5,5,\mathbb{Z})$	$E_{6(6)}(\mathbb{Z})$	$E_{7(7)}(\mathbb{Z})$	$E_{8(8)}(\mathbb{Z})$
D=	10B	9	8	7	6	5	4	3

HOW POWERFUL ARE THE CONSTRAINTS IMPOSED BY SUSY, DUALITY AND UNITARITY ??

The aim is to investigate the exact moduli dependence of low lying terms in the low energy expansion.

Duality relates different regions of moduli space –
connects perturbative and non-perturbative features in a highly nontrivial manner.

e.g.

Four-Graviton Scattering in Type II String Theory

$$A_D(s,t,u;\mu_D) = R^4\, T_D(s,t,u;\mu_D)$$

moduli

R Linearized curvature $\sim k_\mu k_\nu \zeta_{\rho\sigma}$

Symmetric function of Mandelstam invariants s, t, u (with $s+t+u=0$).

Has an expansion in power series of $\sigma_2 = s^2+t^2+u^2$ and $\sigma_3 = s^3+t^3+u^3$.

(non-analytic pieces are essential, but will be ignored here)

$$T_D(s,t,u;\mu_D) = \sum_{p,q} \mathcal{E}^{(D)}_{(p,q)}(\mu_D)\, \sigma_2^p \sigma_3^q \sim s^{2p+3q} + \ldots$$

Coefficients are duality invariant functions of scalar fields (moduli, or coupling constants).

To what extent can we determine these coefficients?

For now focus on the ten-dimensional cases with one modulus:

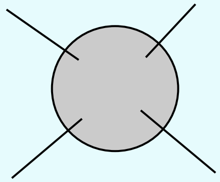
Type IIA: $\Omega = g_A^{-1} = e^{-\phi_A}$

Type IIB: $\Omega = \Omega_1 + i\Omega_2$ $SL(2,\mathbb{Z})$ duality

inverse string coupling constant $\Omega_2 = g_B^{-1} = e^{-\phi_B}$

Boundary data: String Perturbation Theory

TREE-LEVEL: (VIRASORO AMPLITUDE)



Polarisation tensor

dilaton

coupling $g = e^{\phi}$

$$A_0^{(4)}(\epsilon_r, k_r; \phi) = e^{-2\phi}\, R^4\, T_0^{(4)}(s,t,u)$$

$$T_0^{(4)} = \frac{4}{stu}\,\frac{\Gamma(1-\alpha' s)\Gamma(1-\alpha' t)\Gamma(1-\alpha' u)}{\Gamma(1+\alpha' s)\Gamma(1+\alpha' t)\Gamma(1+\alpha' u)}$$

$s^k R^4 \sim d^{2k} R^4$

R^4 $\quad d^4 R^4$ $\quad d^6 R^4$ $\quad d^8 R^4$

Tree-level SUPERGRAVITY

$$= \frac{3}{\sigma_3} + 2\zeta(3)\,\alpha'^3 + \zeta(5)\,\alpha'^5\,\sigma_2 + \frac{2\zeta(3)^2}{3}\,\alpha'^6\,\sigma_3 + \frac{\zeta(7)}{2}\,\alpha'^7\,\sigma_2^2$$

$$+ \frac{2\zeta(3)\zeta(5)}{3}\,\alpha'^8\,\sigma_2\sigma_3 + \frac{\zeta(9)}{4}\,\alpha'^8\,\zeta_2^3 + \frac{2}{27}\left(2\zeta(3)^3 + \zeta(9)\right)\alpha'^9\,\sigma_3^2 + \dots$$

$d^{10} R^4$ $\quad d^{12} R^4$

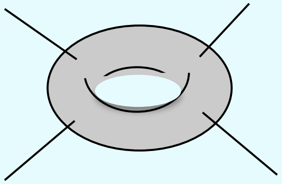
$\sigma_2 = s^2 + t^2 + u^2$

$\sigma_3 = s^3 + t^3 + u^3 = 3stu$

INFINITE SERIES of $d^{2k} R^4$ terms. Coefficients are powers of ζ values with rational coefficients – as in loop amplitudes in quantum field theory

GENUS ONE

moduli $\rho_d \in SO(d,d)/(SO(d)\times SO(d))$



$$A_1^{(4)}(\epsilon_r, k_r; \phi, \rho_d) = \frac{\pi}{16} R^4 \int_{\mathcal{M}_1} \frac{|d\tau|^2}{(\mathrm{Im}\,\tau)^2} \mathcal{B}_1(s,t,u;\tau)\, \Gamma_{d,d,1}(\rho_d;\tau)$$

Genus-one lattice factor for d-torus; moduli ρ_d

Integral over complex structure

$$\mathcal{B}_1(s,t,u;\tau) = \int_{\Sigma^4} \frac{\prod_{i=1}^{i=4} d^2 z}{(\mathrm{Im}\,\tau)^4} \exp\left(-\frac{\alpha'}{2}\sum_{i<j} k_i\cdot k_j\, G(z_i,z_j)\right)$$

Low energy expansion - integrate powers of the genus-one Green function over the torus and over the modulus of the torus – difficult!

e.g $d=0\,,\quad D=10$

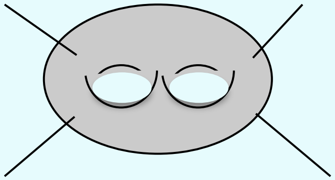
$$A_{1\;an}^{(4)} = \left(\frac{\pi}{3} + 0\,\sigma_2 + \frac{\pi\zeta(3)}{9}\sigma_3 + \ldots\right) R^4$$

(MBG, Russo, Vanhove)

These coefficients look analogous to the tree-level coefficients:

WHAT IS THE CONNECTION BETWEEN THEM??

GENUS TWO :



$$A_2^{(4)}(\epsilon_r, k_r; \phi, \rho_d) = \frac{\pi}{64}\, e^{2\phi}\;R^4 \int_{\mathcal{M}_2} d\mu_2\, \mathcal{B}_2(s,t,u;\Omega)\, \Gamma_{d,d,2}(\rho_d;\Omega)$$

$$\mathcal{B}_2(s,t,u;\Omega) = \int_{\Sigma^4} \frac{|\mathcal{Y}_S|^2}{(\det Y)^2} \exp\left\{-\frac{\alpha'}{2}\sum_{i<j} k_i\cdot k_j\, G(z_i,z_j)\right\}$$

Genus-two Green function

$Sp(4,\mathbb{Z})$-invariant measure $O(s^2)$

Expand in powers of α' :

Lowest-order term $d^4\,R^4$

Next term $d^6\,R^4$

$$A_2^{(4)} = g_s^2\left(\frac{4}{3}\zeta(4)\,\sigma_2 R^4 \quad + \quad 64\int_{\mathcal{M}_2} d\mu_2\,\varphi\,\sigma_3\,R^4 + \ldots\right)$$

Proportional to volume of genus-two moduli space

D'Hoker, Gutperle, Phong

$$\varphi(\Sigma) = -\frac{1}{8}\int_{\Sigma^2} P(x,y)\,G(x,y)$$

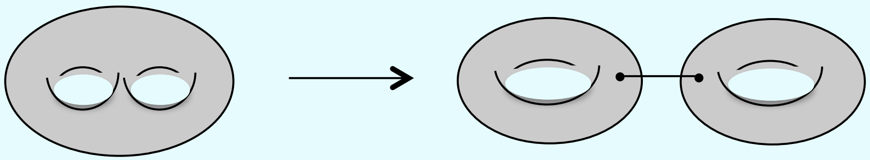
An invariant of genus-h Riemann surface defined by Zhang and Kawazumi.

D'Hoker, MBG

Projection operator proportional to $|\,\omega(x)\,\omega(y)\,|^2$

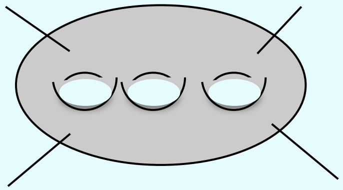
Recently evaluated. D'Hoker, MBG, Pioline, R.Russo

Integral picks up non-zero boundary contribution from the limit in which the genus-two surface degenerates into the union of two genus one surfaces



Result:
$$A_2^{(4)} = g_s^2\Big(\underbrace{\frac{4}{3}\zeta(4)\,\sigma_2 R^4}_{d^4\,R^4} + \underbrace{4\zeta(4)\sigma_3 R^4}_{d^6\,R^4} + \dots \Big)$$

Genus Three and Higher



General issues:

Technical difficulties analysing 3-loops. Last year, Gomez and Mafra constructed the genus-three amplitude using Berkovits' PURE SPINOR FORMALISM. They evaluated the leading low energy behaviour, giving,

Alternative to supermoduli space of RNS formalism

$$A_3^{(4)} = g^4 \left(\frac{4}{27} \zeta(6)\, \underbrace{\sigma_3}_{d^6 R^4} + \dots \right) R^4$$

BUT! There may be a spurious factor of 3

Some Issues at higher genus:

- Problems with singularities in the pure spinor formalism at genus > 4 (for four-graviton amplitude) remain to be resolved.

- New issues for genus > 4 (for four-graviton amplitude) in Ramond-Neveu-Schwarz formalism (integration over super-Riemann surfaces). Superspace is non-projected so cannot express the amplitude as an integral over bosonic moduli. (Donagi, Witten)

NON-PERTURBATIVE EXTENSION

Duality, supersymmetry and unitarity constraints

Focus here on the simplest nontrivial duality group $SL(2,\mathbb{Z})$

Type IIB in D=10 dimensions

The lowest order (BPS) interactions can be determined by (maximal) supersymmetry.
Extending this to higher orders is a (interesting) challenge.

M-THEORY / STRING THEORY DUALITY

Recall:

(A) 11-DIMENSIONAL SUPERGRAVITY on a CIRCLE of radius R_{11}

- Equivalent to TYPE IIA STRING THEORY in ten dimensions with $g_A = \frac{1}{\ell_{11}} R_{11}$

i.e., strong coupling D=10 string $\Leftrightarrow$ large circle D=11 supergravity

11-dim. Planck length

(B) 11-DIMENSIONAL SUPERGRAVITY on a 2-DIMENSIONAL TORUS - $SL(2,\mathbb{Z})$ duality

$$T^2 \times M_9$$

9–dim. Minkowski

T^2 Metric: $$G_{IJ} = \frac{\mathcal{V}^{\frac{1}{2}}}{\Omega_2} \begin{pmatrix} 1 & \Omega_1 \\ \Omega_1 & |\Omega|^2 \end{pmatrix}$$

Volume, $\mathcal{V}$; Complex Structure, $\Omega = \Omega_1 + i\Omega_2$

- Equivalent to TYPE IIB STRING THEORY in $D=9$, radius r_B, coupling constant g_B

$$\mathcal{V} = r_B^{-4/3} e^{\phi_B/3} \qquad \Omega_2 = g_B^{-1}$$

CONSIDER QUANTUM CORRECTIONS TO ELEVEN-DIMENSIONAL SUPERGRAVITY

(Feynman diagrams)

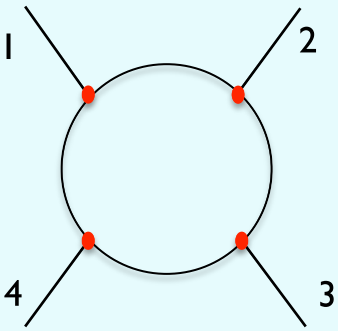
QUANTUM CORRECTIONS TO ELEVEN-DIMENSIONAL SUPERGRAVITY

L-loop Feynman diagrams on $T^2 \times M_9$

ONE LOOP

MBG, Gutperle, Vanhove
Russo, Tseytlin

$$A_1^{(4);sugra} = \quad R^4 \quad \times$$



Scalar box diagram

Interpret as Type IIB string theory in D=9 radius r_B , $g_B = \Omega_2^{-1}$

Sum over winding numbers of loop around 2 cycles of torus

D=10 dimensions $\mathcal{E}_{(0,0)}(\Omega)\, R^4 = \Big(2\zeta(3)\, E_{\frac{3}{2}}(\Omega) + O(r_B^{-1})\Big)\, R^4$
$(r_B \to \infty)$

NON-HOLOMORPHIC EISENSTEIN SERIES

NON-HOLOMORPHIC EISENSTEIN SERIES

$$E_s(\Omega) = \sum_{\gcd(p,q)=1} \frac{\Omega_2^s}{|p+q\Omega|^{2s}} = \sum_{\gamma\in\Gamma_\infty\backslash SL(2,\mathbb{Z})} (\mathrm{Im}\,\gamma\Omega)^s$$

Poincare series – manifest $SL(2,\mathbb{Z})$

Parabolic subgroup

- $SL(2,\mathbb{Z})$ invariant (generalises to higher rank duality groups)

- Solution of LAPLACE EIGENVALUE EQN. (consequence of maximal supersymmetry)

$$\Delta_\Omega\, E_s(\Omega) = s(s-1)\, E_s(\Omega) \qquad \Delta_\Omega = \Omega_2^2(\partial^2_{\Omega_1} + \partial^2_{\Omega_2})$$

- Fourier series $\quad E_s(\Omega) = 2\sum_{k=0}^{\infty} \mathcal{F}_k(\Omega_2)\cos(2\pi i k\Omega_1)\,.$

- ZERO MODE $k=0$ - *TWO POWER-BEHAVED TERMS* (perturbative) :

$$\mathcal{F}_0 = \Omega_2^s + \frac{\sqrt{\pi}\,\Gamma(s-\frac{1}{2})\zeta(2s-1)}{\zeta(2s)\Gamma(s)}\,\Omega_2^{1-s}$$

- NON-ZERO MODES $k>0$ - *D-INSTANTON SUM*

$$\mathcal{F}_k = \frac{2\pi^s}{\zeta(2s)\Gamma(s)}\,|k|^{s-\frac{1}{2}}\,\sigma_{2s-1}(k)\,\Omega_2^{\frac{1}{2}}\,K_{s-\frac{1}{2}}(2\pi|k|\Omega_2)$$
$$\sim \frac{\pi^{s-\frac{1}{2}}}{\zeta(2s)\Gamma(s)}\,|k|^{s-1}\,\sigma_{2s-1}(k)\,e^{-2\pi|k|\Omega_2}$$

measure $\quad \sigma_n(k) = \sum_{p|k} p^n$

One Loop

D=10 dimensions $\mathcal{E}_{(0,0)}(\Omega)\, R^4 = \Big(2\zeta(3)\, E_{\frac{3}{2}}(\Omega) + O(r_B^{-1})\Big)\, R^4$
$(r_B \to \infty)$

$$2\zeta(3)\, g_B^{-\frac{1}{2}}\, E_{\frac{3}{2}}(\Omega) = 2\zeta(3)\, g_B^{-2} + 4\zeta(2)\, g_B^{0} + \ \mathrm{D-instantons}$$

Two perturbative terms: tree-level genus-one

NON-RENORMALISATION BEYOND 1-LOOP FOR R^4

$\frac{1}{2} - BPS$

A NOTE ON THE $AdS_5 \times S^5$ CORRESPONDENCE.

Type IIB STRING THEORY in D=5 Anti de-Sitter space $\Leftrightarrow$ D=4 SU(N) YANG-MILLS on boundary of AdS_5

AdS/CFT dictionary

Inverse string coupling $\longrightarrow$ $\Omega_2 \equiv e^{-\varphi} = \dfrac{4\pi}{g^2_{_{YM}}}$ $\longleftarrow$ YM coupling

AdS length scale $\longrightarrow$ $\dfrac{\alpha'^2}{L^4} = \dfrac{1}{g^2_{YM} N} \equiv \dfrac{1}{\lambda}$ $\longleftarrow$ 't Hooft coupling

Effective R^4 string action $\quad \dfrac{1}{\alpha'} \displaystyle\int d^{10}x \sqrt{-\det G}\, \Omega_2^{-\frac{1}{2}}\, E_{\frac{3}{2}}(\Omega)\, R^4$

$\Leftrightarrow$ Coefficient of gauge invariant Yang-Mills correlator, e.g. $\langle O(x_1) \ldots O(x_4) \rangle$

$N \to \infty$

$1 \ll \lambda \ll N$

$$N^{\frac{1}{2}} \left(2\zeta(3) g_s^{-3/2} + 4\zeta(2) g_s^{\frac{1}{2}} + 2\sqrt{\pi} \sum_{k \neq 0} |k| \sigma_2(k) e^{-2\pi |k|/g_s + 2\pi i k \Omega_1} \right)$$

$$= 2\zeta(3)\, N^2\, \lambda^{-\frac{3}{2}} + 4\zeta(2)\, N^0\, \lambda^{\frac{1}{2}} + 2\sqrt{\pi N} \sum_k |k| \sigma_2(k) e^{-2\pi |k|/g^2_{YM} + 2\pi i k \Omega_1}$$

PLANAR contribution

$\lambda \gg 1$

measure obtained from $SU(N)$ Yang-Mills k-INSTANTON as $N \to \infty$

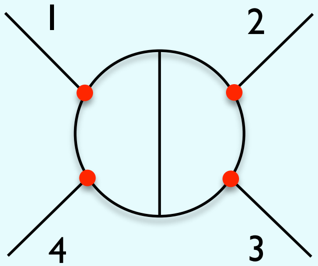
(Dorey, Hollowood, Khoze)

Two Loops

$$A_2^{(4);sugra} = R^4 \times I_2(s,t,u)$$

Bern, Dixon, Dunbar, Perelstein, Rozowsky

- Integrate vertex positions over three lines of skeleton.



Scalar field theory rules

$D^4 R^4$

LEADING TERM IN LOW ENERGY EXPANSION $\quad g_B^{\frac{1}{2}}\,\mathcal{E}_{(1,0)}(\Omega)\,\sigma_2\,R^4$

$$g_B^{\frac{1}{2}}\,\mathcal{E}_{(1,0)}(\Omega)\,\sigma_2\,R^4 = \frac{1}{2} g_B^{\frac{1}{2}}\,E_{\frac{5}{2}}(\Omega) \sim \zeta(5) g_B^{-2} + \frac{4}{3}\zeta(4) g_B^2 + \mathrm{D}-\mathrm{instantons}$$

MBG, Vanhove

Perturbative terms: tree-level genus-two

(no genus-one term)

NON-RENORMALISATION BEYOND 2 LOOPS FOR $D^4 R^4$

$\frac{1}{4} - BPS$

iii) HIGHER ORDER

Next order $\quad \Omega_2^{-1}\,\mathcal{F}_6^{(0)}(\Omega)\,\sigma_3\,\overset{D^6R^4}{\downarrow}R^4 \qquad (\mathcal{F}_6^{(0)}(\Omega)\equiv\mathcal{E}_{(0,1)}(\Omega))$

$1/8\text{-BPS} \quad (\Delta=14\,,\quad n=6\,,\quad u=0)$

Expand integrand to next order in $s,t,u,$ leads to an integral that satisfies

INHOMOGENEOUS LAPLACE EQUATION: (MBG, Vanhove)

$$(\Delta_\Omega-12)\,\mathcal{F}_6^{(0)}(\Omega)=-\Big(2\zeta(3)\,E_{\frac{3}{2}}(\Omega)\Big)^2$$

The square of the coefficient of R^4

The inhomogeneous Laplace equation was obtained by evaluation of two-loop 11-dimensional supergravity compactified on two-torus.

Detailed structure not yet derived in detail from supersymmetry but is based on duality with M-theory :

SOLUTION OF THE INHOMOGENEOUS LAPLACE EQUATION

MBG, Miller, Vanhove

$$(\Delta_\Omega - 12)\, f(\Omega) = -\Big(2\zeta(3)\, E_{\frac{3}{2}}(\Omega)\Big)^2$$

FOURIER SERIES: $f(\Omega) \;=\; \sum_n \widehat{f}_n(\Omega_2)\, e^{2\pi i n\Omega_1}\;.$

EQUATION FOR FOURIER MODES : $(\Omega_2^2\, \partial^2_{\Omega_2} - 12 - 4\pi^2 n^2 \Omega_2^2)\, \widehat{f}(\Omega_2) = S_n(\Omega_2)$ Fourier mode of source

BOUNDARY CONDITIONS : $\widehat{f}_n(\Omega_2) = O(\Omega_2^3)\,, \qquad \Omega_2 \to \infty$ Weak coupling

Weak coupling (TREE LEVEL) power behaviour

$\widehat{f}_n(\Omega_2) = O(\Omega_2^{-2})\,, \qquad \Omega_2 \to 0$ Strong coupling

SUBTLE consequence of $SL(2,\mathbb{Z})$ invariance

These b.c.'s determine a unique solution by fixing the coefficient of the solution of the homogeneous equation, $\alpha_n \sqrt{y}\, K_{\frac{7}{2}}(2\pi|n|y)$, for each value of n

ZERO MODE - four power-behaved terms :

$$\widehat{f}_0(\Omega_2) = \frac{2\,\zeta(3)^2}{3}\,\Omega_2^3 + \frac{4\,\zeta(2)\,\zeta(3)}{3}\,\Omega_2 + \frac{4\,\zeta(4)}{\Omega_2} + \frac{4\,\zeta(6)}{27}\,\Omega_2^{-3} + \sum_{m\neq 0}\widehat{f}_0^{\,m}(\Omega_2)$$

GENUS $\quad 0 \quad 1 \quad 2 \quad 3 \quad$ Non-Perturbative

- ALL PERTURBATIVE CONTRIBUTIONS AGREE WITH EXPLICIT CALCULATIONS

(BUT GENUS 3 string calculation needs RE-CHECKING)

- NON-PERTURBATIVE TERMS

2 X 2 matrix of polynomial coefficients

$$\widehat{f}_0^{\,m}(\Omega_2) = \frac{32\,\pi\,\sigma_2(|m|)^2}{315\,|m|^3}\sum_{i,j=0,1} r^{i,j}(\pi|m|\Omega_2)\,K_i(2\pi|m|\Omega_2)\,K_j(2\pi|m|\Omega_2)$$

Bilinear in $K_0\,,\ K_1$

$\sim e^{-4\pi m\Omega_2}$

$$\Omega_2 \to \infty \qquad \sim e^{-4\pi|m|\Omega_2}\left(\frac{\sigma_2(|m|)^2}{|m|^5\,\Omega_2^2} + O(\Omega_2^{-3})\right)$$

Behaviour suggestive of charge-zero *INSTANTON / ANTI-INSTANTON* pairs.

$$\Omega_2 \to 0 \qquad \sim \frac{945\,\zeta(3)^2\,\zeta(5)}{4\,\pi^5}\,\frac{1}{\Omega_2^2} + O(\log\Omega_2)$$

cancellation of Ω_2^{-3} term by infinite number of "instantons".

NON-ZERO MODES:

$i.j = 0, 1$

2 X 2 matrix of polynomial coefficients

$$\widehat{f}_n(\Omega_2) = \alpha_n \sqrt{\Omega_2}\, K_{\frac{7}{2}}(2\pi|n|\Omega_2) + \sum_{\substack{n_1+n_2=n \\ (n_1,n_2)\neq(0,0)}} M^{ij}_{n_1,n_2}(\pi|n|\Omega_2)\, K_i(2\pi|n_1|\Omega_2)\, K_j(2\pi|n_2|\Omega_2)$$

Constant α_n determined by cancellation of the Ω_2^{-3} term in the $\Omega_2 \to 0$ limit.

$$\underset{\Omega_2>>1}{\sim} e^{-2\pi(|n_1|+|n_2|)\Omega_2}$$

BPS INSTANTON PAIR if $|n_1| + |n_2| = |n| = |n_1 + n_2|$ $\quad (\text{sign}\, n_1 = \text{sign}\, n_2)$

charge = action

"INSTANTON / ANTI-INSTANTON" pair if $|n_1| + |n_2| < |n|$ $\quad (\text{sign}\, n_1 = -\text{sign}\, n_2)$

charge < action

$$\widehat{f}_n(\Omega_2) \underset{\Omega_2>>1}{\sim} e^{-2\pi|n|\Omega_2}\left(8\,\frac{\sigma_2(|n|)}{|n|^{5/2}}\,\zeta(3)\,\Omega_2^{1/2} + O(1)\right) + c\,e^{-2\pi(|n|+1)\Omega_2}(\dots) + \dots$$

- Solution can be expressed as a Poincare series:

$$f(\Omega) = \sum_{\gamma \in \Gamma_\infty \backslash SL(2,\mathbb{Z})} \Phi(\gamma\Omega)$$

where $\Phi(\Omega) = a_0(\Omega_2) + \sum_{n\neq 0} a_n(\Omega_2)\, e^{2\pi i n \Omega_1}$ ($a_n(\Omega_2)$ is linear in $K_0\,,\ K_1$)

- D-instantons contribute with distinctive leading powers of Ω_2 (g^{-1}) – origin not understood in detail.

HIGHER-RANK DUALITY GROUPS

Compactify M-theory on a d-torus to D=11-d dimensions

MBG, Miller, Russo, Vanhove

Pioline

Duality Group $G(\mathbb{Z})$	space-time dimension
1	10A
$SL(2,\mathbb{Z})$	10B
$SL(2,\mathbb{Z})$	9
$SL(3,\mathbb{Z}) \times SL(2,\mathbb{Z})$	8
$SL(5,\mathbb{Z})$	7
$SO(5,5,\mathbb{Z})$	6
$E_{6(6)}(\mathbb{Z})$	5
$E_{7(7)}(\mathbb{Z})$	4
$E_{8(8)}(\mathbb{Z})$	3

Automorphic functions for higher-rank groups ;

Langlands Eisenstein series' associated with maximal parabolic subgroups of G.

$$E^G_{s_1,\dots,s_r}$$

rank r $\quad s_1, \dots, s_r \in \mathbb{C}$

labels associated with nodes of Dynkin diagram

$$E^G_{\frac{3}{2},0,\dots,0}\ R^4 \qquad E^G_{\frac{5}{2},0,\dots,0}\ D^4R^4$$

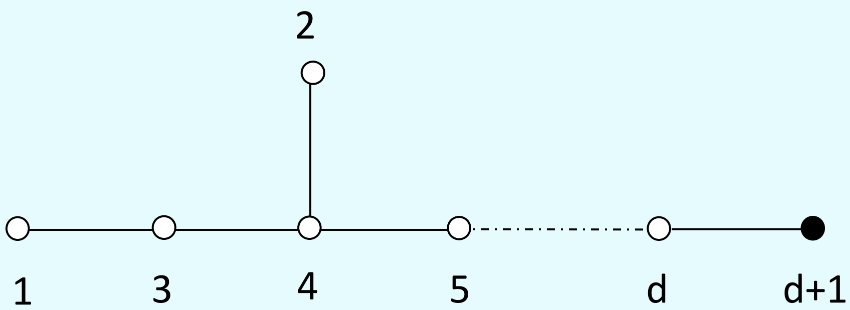
$$\mathcal{E}^G_{(0,1)}\ D^6R^4$$

Satisfies inhomogeneous Laplace equation for G

- Encodes perturbative string results in compactified theories.
- D-INSTANTONS fill out expected fractional BPS orbits – minimal, next-to-minimal, ….

Maximal Parabolic Subgroups and Eisenstein Series

A maximal parabolic subgroup P_β associated with a simple root β



Dynkin diagram for E_{d+1}

$$P_\beta = L_\beta \, U_\beta$$

L_β Levi subgroup obtained by deleting root β

U_β Unipotent radical - the largest normal subgroup consisting of unipotent matrices (upper triangular with unit diagonal).

Maximal parabolic Eisenstein series

$$E^G_{\beta;s} \quad := \sum_{\gamma \in P_\beta(\mathbb{Z}) \backslash G(\mathbb{Z})} e^{\,2\,s\,\langle \omega_\beta , H(\gamma g) \rangle}$$

Cartan

ω_β dual to simple root β

Iwasawa $\quad g \in N \, e^{H(g)} \, K$

COMMENTS:

- Some results on higher derivative interactions for N < 8 SUSY e.g. Tourquine, Vanhove and in open string theory.

String theory is free of ultraviolet divergences at all orders.
All supergravity field theory Feynman diagrams are packaged into a single world-sheet diagram.

FANTASY: DETERMINE PROPERTIES OF SUPERGRAVITY FEYNMAN DIAGRAMS BY SUITABLE LIMIT OF STRING THEORY DIAGRAMS