Feynman Integrals, Elliptic polylogarithms and mixed Hodge structures

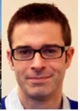
Pierre Vanhove

New Geometric structures meeting Oxford University

Septembre 23, 2014

based on [arXiv:1309.5865], [arXiv:1406.2664] and work in progress

Spencer Bloch, Matt Kerr

There are many technics to compute amplitudes in field theory

- On-shell (generalized) unitarity
- On-shell recursion methods
- twistor geometry, Graßmanian, Symbol,...
- Dual conformal invariance
- Infra-red behaviour (inverse soft-factors, ...)
- amplitude relations,
- String theory,...

These methods indicate that amplitudes have simpler structures than the diagrammatic from Feynman rules suggest

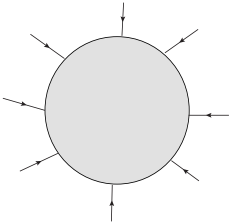
The questions are:
what are the basic building blocks of the amplitudes?
Can the amplitudes be expressed in a basis of integrals functions?

Part I

One-loop amplitudes

The one-loop amplitude

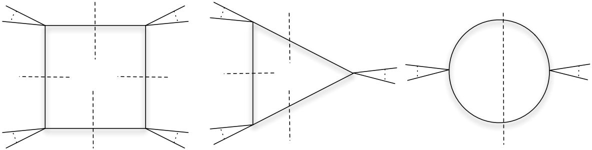
- In four dimensions any one-loop amplitude can be expressed on a basis of integral functions [Bern, Dixon,Kosower]



$$= \sum_r c_r \, \text{Integral}_r + \text{Rational terms}$$

The one-loop amplitude

- The integral functions are the box, triangle, bubble, tadpole



- The integrals functions are given by dilogarithms and logarithms

$$\text{Boxes, Triangles} \sim \mathrm{Li}_2(z) = -\int_0^z \log(1-t)\, d\log t$$

$$\text{Bubble} \sim \log(1-z) = \int_0^z d\log(1-t)$$

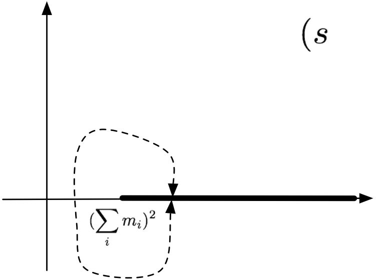
The one-loop amplitude

This allows to characterize in a simple way one-loop amplitudes in various gauge theory

- Only boxes for $\mathcal{N} = 4$ SYM for one-loop graph
- No triangle property of $\mathcal{N} = 8$ SUGRA [Bern, Carrasco, Forde, Ita, Johansson; Arkani-hamed, Cachazo, Kaplan; Bjerrum-Bohr, Vanhove]
- Only box for QED multi-photon amplitudes with $n \geqslant 8$ photons [Badger, Bjerrum-Bohr, Vanhove]

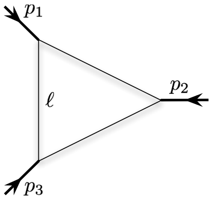
Monodromies, periods

- Amplitudes are multivalued quantities in the complex energy plane with monodromies around the branch cuts for particle production



- They satisfy differential equation with respect to its parameters : kinematic invariants s_{ij}, internal masses m_i, ...
- monodromies with differential equations : typical of periods integrals

A one-loop example I



We consider the 3-mass triangle $p_1 + p_2 + p_3 = 0$ and $p_i^2 \neq 0$

$$I_\triangleright(p_1^2, p_2^2, p_3^2) = \int_{\mathbb{R}^{1,3}} \frac{d^4\ell}{\ell^2(\ell+p_1)^2(\ell-p_3)^2}$$

Which can be represented as

$$I_\triangleright = \int_{\substack{x\geqslant 0\\ y\geqslant 0}} \frac{dxdy}{(p_1^2 x + p_2^2 y + p_3^2)(xy + x + y)}$$

A one-loop example II

and evaluated as

$$I_{\triangleright} = \frac{D(z)}{\left(p_1^4 + p_2^4 + p_3^4 - (p_1^2 p_2^2 + p_1^2 p_3^2 + p_2^2 p_3^2)\right)^{\frac{1}{2}}}$$

z and $\bar{z}$ roots of $(1 - x)(p_3^2 - x p_1^2) + p_2^2 x = 0$

- Single-valued Bloch-Wigner dilogarithm for $z \in \mathbb{C}\backslash\{0, 1\}$

$$D(z) = \mathfrak{Im}(Li_2(z)) + \arg(1 - z) \log |z|$$

- The permutation of the 3 masses: $\{z, 1 - \bar{z}, \frac{1}{\bar{z}}, 1 - \frac{1}{z}, \frac{1}{1-z}, -\frac{\bar{z}}{1-\bar{z}}\}$ this set is left invariant by the $D(z)$
- The integral has branch cuts arising from the square root since $D(z)$ is analytic

The triangle graph motive I

$$I_\triangleright = \int_{\substack{x\geqslant 0\\ y\geqslant 0}} \frac{dxdy}{(p_1^2x+p_2^2y+p_3^2)(xy+x+y)}$$

The integral is defined over the domain $\Delta = \{[x,y,z]\in\mathbb{P}^2, x,y,z\geqslant 0\}$ and the denominator is the quadric

$$C_\triangleright := (p_1^2x+p_2^2y+p_3^2z)(xy+xz+yz)$$

Let $L=\{x=0\}\cup\{y=0\}\cup\{z=0\}$ and $D=\{x+y+z=0\}\cup C_\triangleright$

$$\frac{dxdy}{(p_1^2x+p_2^2y+p_3^2)(xy+x+y)} \in H := H^2(\mathbb{P}^2-D, L\backslash(L\cup C_\triangleright)\cap L, \mathbb{Q})$$

We need to consider the relative cohomology because the domain Δ is not in $H_2(\mathbb{P}^2-D)$ because $\partial\Delta\cap\neq\emptyset$

The triangle graph motive II

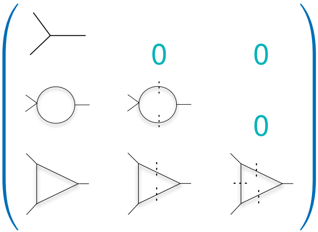
Since $\partial\Delta \cap C_{\triangleright} = \{[1,0,0],[0,1,0],[0,0,1]\}$ one needs to perform a blow-up these 3 points.

One can define a mixed Tate Hodge structure [Bloch, Kreimer] with weight $W_0H \subset W_2H \subset W_4H$ and grading

$$gr_0^W H = \mathbb{Q}(0), \qquad gr_2^W H = \mathbb{Q}(-1)^5, \qquad gr_4^W H = \mathbb{Q}(-2)$$

The Hodge matrix and unitarity

$$\begin{pmatrix} 1 & 0 & 0 \\ -Li_1(z) & 2i\pi & 0 \\ -Li_2(z) & 2i\pi \log z & (2i\pi)^2 \end{pmatrix}$$



- The construction is valid for all one-loop amplitudes in four dimensions

Part II

Loop amplitudes

Feynman parametrization

- Typically form of the Feynman parametrization of a graph Γ
- A Feynman graph with L loops and n propagators

$$I_\Gamma \propto \int_0^\infty \delta(1-\sum_{i=1}^n x_i)\,\frac{\mathcal{U}^{n-(L+1)\frac{D}{2}}}{(\mathcal{U}\sum_i m_i^2 x_i - \mathcal{F})^{n-L\frac{D}{2}}}\prod_{i=1}^n dx_i$$

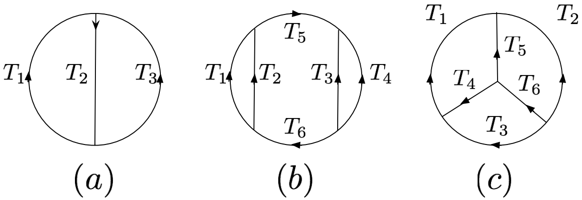
- $\mathcal{U}$ and $\mathcal{F}$ are the Symanzik polynomials [Itzykson, Zuber]
- $\mathcal{U}$ is of degree L and $\mathcal{F}$ of degree $L+1$

Feynman parametrization and world-line formalism

- Rewrite the integral as

$$I_\Gamma \propto \int_0^\infty \frac{\delta(1-\sum_{i=1}^n x_i)}{(\sum_i m_i^2 x_i - \widehat{\mathcal{F}})^{n-L\frac{D}{2}}} \frac{\prod_{i=1}^n dx_i}{\mathcal{U}^{\frac{D}{2}}}$$

- $\mathcal{U} = \det \Omega$ is the determinant of the period matrix of the graph



- The period matrix of integral of homology vectors v_i on oriented loops C_i

$$\Omega_{ij} = \oint_{C_i} v_j$$

Feynman parametrization and world-line formalism

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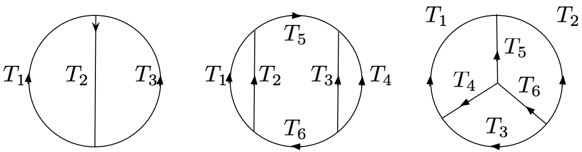

(a) (b) (c)

$$\Omega_{2(a)} = \begin{pmatrix} T_1 + T_3 & T_3 \\ T_3 & T_2 + T_3 \end{pmatrix}$$

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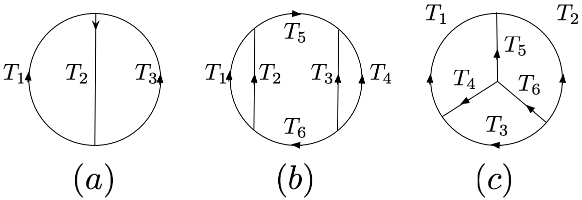

$$\Omega_{3(b)} = \begin{pmatrix} T_1+T_2 & T_2 & 0 \\ T_2 & T_2+T_3+T_5+T_6 & T_3 \\ 0 & T_3 & T_3+T_4 \end{pmatrix}$$

Feynman parametrization and world-line formalism

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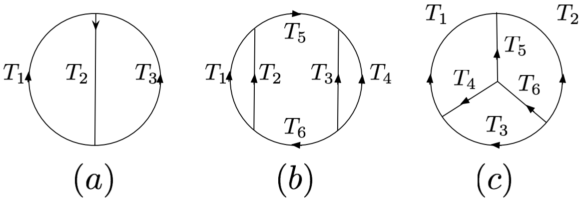


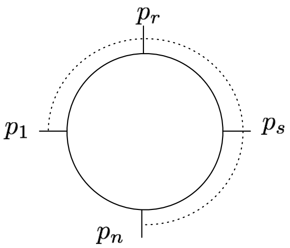
$$\Omega_{3(c)} = \begin{pmatrix} T_1+T_4+T_5 & T_5 & T_4 \\ T_5 & T_2+T_5+T_6 & T_6 \\ T_4 & T_6 & T_3+T_4+T_6 \end{pmatrix}$$

Feynman parametrization and world-line formalism

- Rewrite the integral as

$$I_\Gamma \propto \int_0^\infty \frac{\delta(1-\sum_{i=1}^n x_i)}{(\sum_i m_i^2 x_i - \widehat{\mathcal{F}})^{n-L\frac{D}{2}}} \frac{\prod_{i=1}^n dx_i}{\mathcal{U}^{\frac{D}{2}}}$$

- $\widehat{\mathcal{F}} = \sum_{1\leqslant r<s\leqslant n} k_r \cdot k_s G(x_r, x_s; \Omega)$ sum of Green's function



$$G^{1-loop}(x_r, x_s; L) = -\frac{1}{2}|x_s - x_r| + \frac{1}{2}\frac{(x_r-x_s)^2}{T}.$$

Feynman parametrization and world-line formalism

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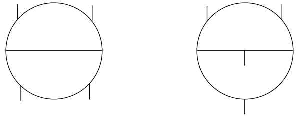

$$G^{2-loop}_{same\ \ line}(x_r, x_s; \Omega_2) = -\frac{1}{2}|x_s - x_r| + \frac{T_2+T_3}{2}\frac{(x_s-x_r)^2}{(T_1T_2+T_1T_3+T_2T_3)}$$

$$G^{2-loop}_{diff\ \ line}(x_r, x_s; \Omega_2) = -\frac{1}{2}(x_r + x_s) + \frac{T_3(x_r+x_s)^2 + T_2x_r^2 + T_1x_s^2}{2(T_1T_2+T_1T_3+T_2T_3)}$$

Periods

- A Feynman graph with L loops and n propagators

$$I_\Gamma \propto \int_0^\infty \delta(1-\sum_{i=1}^n x_i)\, \frac{\mathcal{U}^{n-(L+1)\frac{D}{2}}}{(\mathcal{U}\sum_i m_i^2 x_i - \mathcal{F})^{n-L\frac{D}{2}}} \prod_{i=1}^n dx_i$$

- [Kontsevich, Zagier] define periods are follows.

$\mathcal{P} \in \mathbb{C}$ is the ring of periods, is $z \in \mathcal{P}$ if $\Re\mathrm{e}(z)$ and $\Im\mathrm{m}(z)$ are of the forms

$$\int_{\Delta\in\mathbb{R}^n} \frac{f(x_i)}{g(x_i)} \prod_{i=1}^n dx_i < \infty$$

with $f, g \in \mathbb{Z}[x_1, \cdots, x_n]$ and Δ is algebraically defined by polynomial inequalities and equalities.

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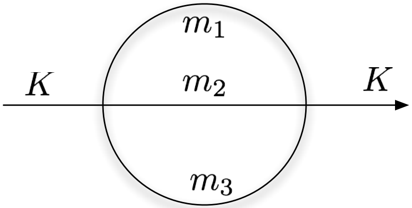
with $f, g \in \mathbb{Z}[x_1, \cdots, x_n]$ and Δ is algebraically defined by polynomial inequalities and equalities.

- Problem for Feynman graphs $\partial\Delta \cap \{g(x_i) = 0\} \neq \emptyset$
- Generally the domain of integration is not closed $\partial\Delta \neq \emptyset$
- Need to consider the relative cohomology and perform blow-ups [Bloch,Esnault,Kreimer]

Part III

The banana graphs

the two-loop sunset integral



We consider the two-loop sunset integral in two Euclidean dimensions given by

$$\mathcal{J}^2_\ominus \propto \int_{\mathbb{R}^4} \frac{d^2\ell_1 d^2\ell_2}{(\ell_1^2+m_1^2)(\ell_2^2+m_2^2)((\ell_1+\ell_2-K)^2+m_3^2)}$$

- Related to $D = 4$ by dimension shifting formula [Tarasov; Baikov; Lee]
- Expression given in term of elliptic function [Laporta, Remiddi; Adams, Bogner, Weinzeirl]

the two-loop sunset integral

If you are interested by the value of the sunset integral in four dimensions: use the relation between Feynman integral in various dimensions [Laporta]

$$\mathcal{J}_{\ominus}^{4-2\epsilon}(K^2, m^2) = 16\pi^{4-2\epsilon}\Gamma(1+\epsilon)^2 \left(\frac{m^2}{\mu^2}\right)^{1-2\epsilon} \left(\frac{a_2}{\epsilon^2} + \frac{a_1}{\epsilon} + a_0 + O(\epsilon)\right)$$

$$\begin{aligned} a_2 &= -\frac{3}{8} \\ a_1 &= \frac{18-t}{32} \\ a_0 &= \frac{(t-1)(t-9)}{12}\left(1+(t+3)\frac{d}{dt}\right)\mathcal{J}_{\ominus}^{2}(t) + \frac{13t-72}{128}\,. \end{aligned}$$

the two-loop sunset integral

The Feynman parametrisation is given by

$$\mathcal{J}^2_\ominus = \int_{\substack{x\geqslant 0\\ y\geqslant 0}} \frac{dxdy}{(m_1^2 x + m_2^2 y + m_3^2)(x+y+xy) - K^2 xy} = \int_{\mathcal{D}} \omega\,.$$

- One has the remarkable representation

$$\mathcal{J}^2_\ominus = 2^2 \int_0^\infty x\, I_0(\sqrt{t}x) \prod_{i=1}^{3} K_0(m_i x)\, dx$$

the two-loop sunset integral

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- The sunset integral is the integration of the 2-form ω

$$\omega = \frac{zdx\wedge dy + xdy\wedge dz + ydz\wedge dx}{A_{\ominus}(x,y,z)} \in H^2(\mathbb{P}^2 - \mathcal{E}_{K^2})$$

- The graph is based on the elliptic curve $\mathcal{E}_{K^2} : A_{\ominus}(x,y,z) = 0$

$$A_{\ominus}(x,y,z) := (m_1^2 x + m_2^2 y + m_3^2 z)(xz + xy + yz) - K^2 xyz\,.$$

the two-loop sunset integral

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- The domain of integration $\mathcal{D}$ is

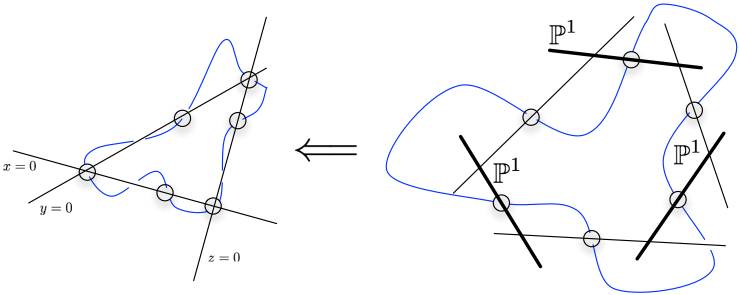
$$\mathcal{D} := \{[x:y:z]\in\mathbb{P}^2 | x\geqslant 0, y\geqslant 0, z\geqslant 0\}$$

the sunset graph mixed Hodge structure

- The elliptic curve intersects the domain of integration $\mathcal{D}$

$$\mathcal{D} \cap \{A_\ominus(x,y,z)=0\} = \{[1:0:0],[0:1:0],[0:0:1]\}$$

- We need to blow-up work in $\mathbb{P}^2 - \mathcal{E}_{K^2}$



the sunset graph mixed Hodge structure

- The domain of integration $\mathcal{D} \notin H_2(\mathbb{P}^2 - \mathcal{E}_{K^2})$ because $\partial\mathcal{D} \neq \emptyset$
- Need to pass to the relative cohomology
- If $P \to \mathbb{P}^2$ is the blow-up and $\hat{\mathcal{E}}_{K^2}$ is the strict transform of $\mathcal{E}_{K^2}$
- Then in P we have resolved the two problems

$$\mathcal{D} \cap \hat{\mathcal{E}}_{K^2} = \emptyset; \qquad \mathcal{D} \in H_2(P - \hat{\mathcal{E}}_{K^2}, \mathfrak{h} - (\mathfrak{h} \cap \hat{\mathcal{E}}_{K^2}))$$

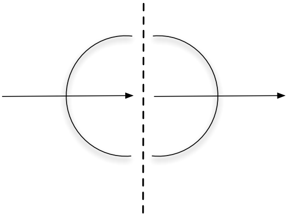
- We have a variation (with respect to K^2) of Hodge structures

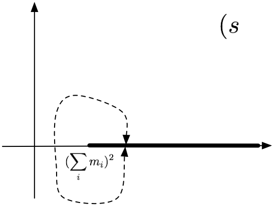
$$H^2_{K^2} := H^2(P - \hat{\mathcal{E}}_{K^2}, \mathfrak{h} - (\mathfrak{h} \cap \hat{\mathcal{E}}_{K^2}))$$

- The sunset integral is a period of the mixed Hodge structure $H^2_{K^2}$

[Bloch, Esnault, Kreimer; Müller-Stach, Weinzeirl, Zayadeh]

Special values



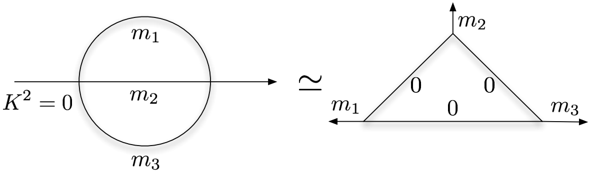


$$\mathcal{J}^2_{\ominus}(K^2, m_i^2) = \int_{\substack{x\geqslant 0\\ y\geqslant 0}} \frac{dxdy}{(m_1^2 x + m_2^2 y + m_3^2)(x + y + xy) - K^2 xy}$$

- Branch cut at the 3-particle threshold $K^2 = (m_1 + m_2 + m_3)^2$ so

$$K^2 \in \mathbb{C} \backslash [(m_1 + m_2 + m_3)^2, \infty[$$

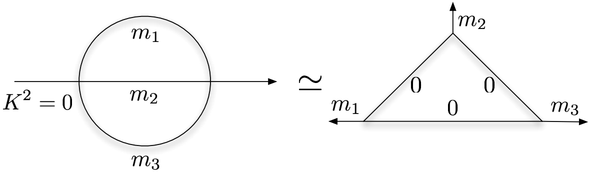
Special values



- At the special fibers $K^2 = 0$ the sunset equal a massive one-loop triangle graph (dual graph transformation)

$$\mathcal{I}^2_{\ominus}(0, m_i^2) \propto \frac{1}{m_1^2} \frac{D(z)}{z - \bar{z}} \qquad (1-x)(m_3^2 - m_1^2 x) - m_2^2 x = m_1^2 (x - z)(x - \bar{z})$$

Special values



- At the special fibers $K^2 = 0$ the sunset equal a massive one-loop triangle graph (dual graph transformation)
- For equal masses $m_i = m$ then $z = \zeta_6$ such that $(\zeta_6)^6 = 1$

$$\mathcal{I}^2_{\ominus}(0) \propto \frac{1}{m^2} \frac{D(\zeta_6)}{\mathfrak{Im}(\zeta_6)}$$

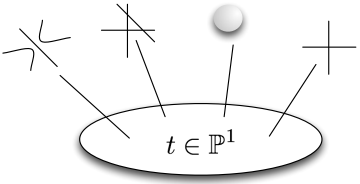
the elliptic curve of the sunset integral

- With all mass equal $m_i = m$ and $t = K^2/m^2$ the integral reduces to

$$\mathcal{J}_{\ominus}(t) = \frac{1}{m^2}\int_0^\infty\int_0^\infty \frac{dxdy}{(x+y+1)(x+y+xy)-txy}\,.$$

$$\mathcal{E}_t:\ (x+y+1)(x+y+xy)-txy = 0\,.$$

- Special values
  - At $t = 0$, $t = 1$ and $t = +\infty$ the elliptic curve factorizes.
  - At $t = 9$ we have the 3-particle the threshold $t \in \mathbb{C}\backslash[9,+\infty[$.



the elliptic curve of the sunset integral

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$$\mathcal{E}_t : \; (x+y+1)(x+y+xy) - txy = 0\,.$$

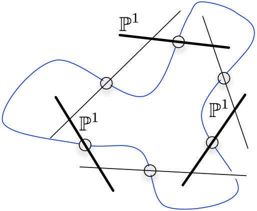
- Family of elliptic curve surface with 4 singular fibers leads to a K_3 pencil

$$\begin{array}{ccc} \mathcal{E}_t & \longrightarrow & \overline{\mathcal{E}}_t \\ f \big\downarrow & & \bar{f} \big\downarrow \\ X_1(6) & \longrightarrow & X_1(6) \cup \{cusps\} \end{array}$$

- This is a universal family of $X_1(6)$ modular curves with a point of order 6

The sunset integral and the motive

$$\begin{array}{ccc} \mathcal{E}_t & \longrightarrow & \overline{\mathcal{E}}_t \\ f \downarrow & & \bar{f} \downarrow \\ X_1(6) & \longrightarrow & X_1(6) \cup \{cusps\} \end{array}$$



- Elliptic local system V on $X_1(6)$ with fibre $\mathbb{Q}^2$
- For $s_1, s_2 \in \mathfrak{h} \cap \mathcal{E}_t$ then the divisor $s_1 - s_2$ on $\mathcal{E}_t$ is of torsion of order 6
- therefore $H_t^2 = \mathbb{Q}(0)^3 \oplus \hat{H}_t^2$

$$0 \longrightarrow \mathbb{Q}(0) \longrightarrow \hat{H}_t^2 \longrightarrow H^1(\mathcal{E}_t, \mathbb{Q}(-1)) \to 0$$

the picard-fuchs equation of the sunset integral

$$\mathcal{E}_t : (x+y+1)(x+y+xy) - txy = 0$$

- The picard-fuchs operator is

$$L_t = \frac{d}{dt}\left(t(t-1)(t-9)\frac{d}{dt}\right) + (t-3)$$

- Acting on the integral we have

$$L_t \mathcal{I}^2_\ominus(t) = \int_{\mathcal{D}} d\beta = -\int_{\partial\mathcal{D}} \beta \neq 0$$

- We find using the Bessel integral representation $\int_0^\infty x I_0(\sqrt{t}x) K_0(x)^3 dx$

$$\frac{d}{dt}\left(t(t-1)(t-9)\frac{d\mathcal{I}_\ominus(t)}{dt}\right) + (t-3)\mathcal{I}_\ominus(t) = -6$$

the sunset integral as an elliptic dilogarithm

- The Hauptmodul t is given by [Zagier; Stienstra]

$$t = 9 + 72\frac{\eta(2\tau)}{\eta(3\tau)}\left(\frac{\eta(6\tau)}{\eta(\tau)}\right)^5$$

- The period ϖ_r and ϖ_c, with $q := \exp(2i\pi\tau(t))$ are given by

$$\varpi_r \sim \frac{\eta(\tau)^6\eta(6\tau)}{\eta(2\tau)^3\eta(3\tau)^2}; \qquad \varpi_c = \tau\,\varpi_r$$

the sunset integral as an elliptic dilogarithm

This leads to the solutions of the PF equation [Bloch, Vanhove]

$$\mathcal{I}^2_{\ominus}(t) = i\pi\varpi_r(t)(1-2\tau) - \frac{6\varpi_r(t)}{\pi} E_{\ominus}(\tau)\,,$$

$$E_{\ominus}(\tau) = \frac{D(\zeta_6)}{\Im\mathrm{m}(\zeta_6)} - \frac{1}{2i}\sum_{n\geqslant 0}\left(\mathrm{Li}_2(q^n\zeta_6) + \mathrm{Li}_2\left(q^n\zeta_6^2\right) - \mathrm{Li}_2\left(q^n\zeta_6^4\right) - \mathrm{Li}_2\left(q^n\zeta_6^5\right)\right)$$

- We have $\mathrm{Li}_2(x)$ and not the Bloch-Wigner $D(x)$

the sunset integral as an elliptic dilogarithm

This leads to the solutions of the PF equation [Bloch, Vanhove]

$$\mathcal{J}^2_{\ominus}(t) = i\pi\varpi_r(t)(1-2\tau) - \frac{6\varpi_r(t)}{\pi}\, E_{\ominus}(\tau)\,,$$

$$E_{\ominus}(\tau) = \frac{1}{2}\sum_{n\neq 0}\frac{\psi_2(n)}{n^2}\,\frac{1}{1-q^n}$$

- $\psi_2(n)$ is an odd mod 6 character

$$\psi_2(n) = \begin{cases} 1 & \textit{for } n \equiv 1 \mod 6 \\ -1 & \textit{for } n \equiv 5 \mod 6 \end{cases}$$

The sunset integral and the motive

- The integral is given by

$$\mathcal{I}^2_{\ominus}(t) = \int_0^\infty \int_0^\infty \frac{dxdy}{(x+y+1)(x+y+xy)-txy}$$

- The 2-form has only log-pole on $\mathcal{E}_t$ and there is a residue 1-form

$$\mathcal{I}^2_{\ominus}(t) = periods + \varpi_r \left\langle \epsilon_1\tau + \epsilon_2, \int d\tau \sum_{(m,n)\neq(0,0)} \frac{\psi_2(n)(\epsilon_1\tau+\epsilon_2)}{(m+n\tau)^3} \right\rangle$$

- Character $\psi : Lattice(\mathcal{E}_t) \to S^1$. Pairing $\langle \epsilon_1, \epsilon_2 \rangle = -\langle \epsilon_2, \epsilon_1 \rangle = 2i\pi$
- The amplitude integral is *not* the regulator map which involves a real projection $r : K_2(\mathcal{E}_t) \to H^1(\mathcal{E}_t, \mathbb{R})$
- The amplitude is multivalued in t whereas the regulator is single-valued

The sunset integral and the motive

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- The regulator is an Eichler integral

$$\mathcal{J}^2_{\ominus}(t) = periods + \varpi_r \int_\tau^{i\infty} \sum_{(m,n)\neq(0,0)} \frac{\psi_2(n)(\tau - x)}{(m+nx)^3}\, dx$$

The sunset integral and the motive

- The integral is given by

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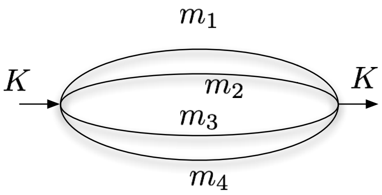
- The regulator is an Eichler integral

$$\mathcal{J}^2_{\ominus}(t) = periods + \varpi_1 \sum_{(m,n)\neq(0,0)} \frac{\psi_2(n)}{n^2(m+n\tau)}$$

three-loop banana graph: integral



We look at the 3-loop banana graph in $D = 2$ dimensions

- The Feynman parametrisation is given by

$$I^2_{\circleddash}(m_i; K^2) = \int_{x_i \geqslant 0} \frac{1}{(m_4^2 + \sum_{i=1}^3 m_i^2 x_i)(1 + \sum_{i=1}^3 x_i^{-1}) - K^2} \prod_{i=1}^3 \frac{dx_i}{x_i}$$

three-loop banana graph: differential equation

- The geometry of the 3-loop banana graph is a K_3 surface (Shioda-Inose family for $\Gamma_1(6)^{+3}$) with Picard number 19 and discriminant of Picard lattice is 6

$$(m_4^2 + \sum_{i=1}^{3} m_i^2 x_i)(1 + \sum_{i=1}^{3} x_i^{-1}) \prod_{i=1}^{3} x_i - t \prod_{i=1}^{3} x_i = 0$$

- The all equal mass case with $t = K^2/m^2$ satisfies the Picard-Fuchs equation [vanhove]

$$\Big(t^2(t-4)(t-16)\frac{d^3}{dt^3} + 6t(t^2-15t+32)\frac{d^2}{dt^2} + (7t^2-68t+64)\frac{d}{dt} + t - 4 \Big) J^2_{\ominus}(t) = -4!$$

- One miracle is that this picard-fuchs operator is the symmetric square of the picard-fuchs operator for the sunset graph [Verrill]

three-loop banana graph: solution

- It is immediate to use the Wronskian method to solve the differential equation [Bloch, Kerr, Vanhove]

$$m^2\, I^2_{\ominus}(t) = 40\pi^2 \log(q)\, \varpi_1(\tau)$$
$$-48\varpi_1(\tau)\left(24\mathcal{L}i_3(\tau,\zeta_6) + 21\,\mathcal{L}i_3(\tau,\zeta_6^2) + 8\mathcal{L}i_3(\tau,\zeta_6^3) + 7\mathcal{L}i_3(\tau,1)\right)$$

with $\mathcal{L}i_3(\tau,z)$ [Zagier; Beilinson, Levin]

$$\mathcal{L}i_3(\tau,z) := \mathrm{Li}_3(z) + \sum_{n\geqslant 1}\left(\mathrm{Li}_3(q^n z) + \mathrm{Li}_3\left(q^n z^{-1}\right)\right)$$
$$-\left(-\frac{1}{12}\log(z)^3 + \frac{1}{24}\log(q)\log(z)^2 - \frac{1}{720}(\log(q))^3\right).$$

three-loop banana graph: solution

- Which can be written using as an Eisenstein series

$$m^2\, I^2_{\ominus}(t(\tau)) = \varpi_1(\tau)\, \left(-4(\log q)^3 + \frac{1}{2}\sum_{n\neq 0} \frac{\psi_3(n)}{n^3}\,\frac{1+q^n}{1-q^n}\right)$$

where $\psi_3(n)$ is an even mod 6 character

- Arising from the regulator for $\mathrm{Sym}^2 H^1(\mathcal{E}_{\ominus}(t))$ with $dz = \epsilon_1\tau + \epsilon_2$

$$\varpi_1\, \left\langle dz^2, \int \sum_{m,n} \frac{d\tau dz^2\, \psi_3(n)}{(m+n\tau)^4} \right\rangle$$

- Again this is given by the Eichler integral

$$\mathrm{period} + \varpi_1 \int_{\tau}^{i\infty} \sum_{m,n} \frac{(x-\tau)^2\, \psi_3(n)}{(m+nx)^4}\, dx$$

Part IV

Higher-loop banana amplitudes

Higher-order bananas I

The integral we have been discussing are given by

$$I_n^2 = \int_{x_i \geqslant 0} \frac{1}{(1+\sum_{i=1}^n x_i)(1+\sum_{i=1}^n x_i^{-1}) - t} \prod_{i=1}^n \frac{dx_i}{x_i}$$

They are given by the following Bessel integral representation

$$I_n^2(t) = 2^{n-1} \int_0^\infty x\, I_0(\sqrt{t}x) K_0(x)^n \, dx$$

The Bessel function $K_0(x)$ satisfies a differential equation that implies that

$$\left(\sum_{r=0}^{n+1} p_r(x) \left(x \frac{d}{dx}\right)^r\right) K_0(x)^n = 0$$

This implies that the banana integral satisfies a differential equation

Higher-order bananas II

$$\left(\sum_{r=0}^{n+1} \tilde{p}_r(t) \left(t\frac{d}{dt}\right)^r\right) I_n^2(t) = 0$$

Because $p_r(x)$ are polynomials in x^2 without constant terms [Borwein, Salvy] then one can factorizes a $(td/dt)^2$ operator and the differential operator for the $n-1$-loop banana graph is of order the number of loops

$$\left(\sum_{r=0}^{n-1} q_r(t) \left(t\frac{d}{dt}\right)^r\right) I_n^2(t) = S_n + \tilde{S}_n \log(t)$$

Since the banana integral is regular at $t=0$ then $\tilde{S}_n = 0$ and the differential equation is

Higher-order bananas III

$$\left(\sum_{r=0}^{n-1} q_r(t)\left(t\frac{d}{dt}\right)^r\right) I_n^2(t) = S_n$$

- with [vanhove]

$$\begin{aligned} q_{n-1}(t) &= t^{\lfloor \frac{n}{2}\rfloor+\eta(n)} \prod_{i=0}^{\lfloor \frac{n}{2}\rfloor}(t-(n-2i)^2) \\ q_{n-2}(t) &= \frac{n-1}{2}\frac{dq_{n-1}(t)}{dt} \\ q_0(t) &= t-n\,. \end{aligned}$$

- with $\eta(n) = 0$ if $n \equiv 1 \mod 2$ and 1 if $n \equiv 0 \mod 2$.
- The inhomogeneous term $S_n = -n!$

Higher-order bananas

- one-loop banana bubble

$$(t-2)\, f(t) + t(t-4)\, f^{(1)}(t) = -2!$$

- Solution are logarithms

$$I_\circ^2(m_1, m_2, K^2) = \frac{\log(z^+) - \log(z^-)}{\sqrt{\Delta}}$$

- $z^\pm$ and Δ are roots and discriminant of the equation
$$(m_1^2 x + m_2^2)(1+x) - K^2 x = m_1^2 (x - z^+)(x - z^-) = 0$$

Higher-order bananas

- four-loop banana bubble

$$(t-5)\,f\,(t)+(3t-5)(5t-57)f^{(1)}(t)$$

$$+\left(25\,t^3-518\,t^2+1839\,t-450\right)\,f^{(2)}(t)$$

$$+\left(10\,t^4-280\,t^3+1554\,t^2-900\,t\right)\,f^{(3)}(t)$$

$$+t^2(t-25)(t-1)(t-9)f^{(4)}(t)=-5!$$

Higher-order bananas

- five-loop banana bubble

$$(t-6)\,f(t) + \left(31\,t^2 - 516\,t + 1020\right) f^{(1)}(t)$$

$$+ \left(90\,t^3 - 2436\,t^2 + 12468\,t - 6912\right) f^{(2)}(t)$$

$$+ \left(65\,t^4 - 2408\,t^3 + 19836\,t^2 - 27648\,t\right) f^{(3)}(t)$$

$$+ \left(15\,t^5 - 700\,t^4 + 7840\,t^3 - 17280\,t^2\right) f^{(4)}(t)$$

$$+t^3(t-36)(t-4)(t-16) f^{(5)}(t) = -6!$$

outlook

- Connection between the Feynman parametrization and the world-line formalism : Possible treatment of field theory graph a la string theory
- Amplitudes are multivalued functions given by motivic periods. The Hodge matrix corresponds to unitarity (discontinuity, cut).

outlook

- At special values $t = 1$ the integrals are *pure* periods related to values of L-functions in the critical band [Broadhurst]

$$I_{\ominus}(1) = \frac{4\pi}{\sqrt{15}} L(K_3, 2)$$

The new form for the $L(K_3, s)$ [Peeters, Top, van der Vlugt]

$$f(\tau) = \eta(\tau)\eta(3\tau)\eta(5\tau)\eta(15\tau) \sum_{m,n} q^{m^2+mn+4n^2} \in S_3\left(15, \left(\frac{\cdot}{15}\right)\right)$$

1.3 Multiplicative modular forms and eta products

For levels $N < 16$, there are precisely 15 multiplicative modular forms that are products of eta values. Here they are listed with notes on quantum field theory (QFT):

Broadhurst talk @ IHES

form	weight	level	QFT
$\eta_1^2\eta_{11}^2$	2	11	
$\eta_1\eta_2\eta_7\eta_{14}$	2	14	
$\eta_1\eta_3\eta_5\eta_{15}$	2	15	
$\eta_1^3\eta_7^3$	3	7	BS
$\eta_1^2\eta_2\eta_4\eta_8^2$	3	8	BS
$\eta_2^3\eta_6^3$	3	12	BS + BBBG + BV : Sections 3 and 4
$\eta_1^4\eta_5^4$	4	5	BS
$\eta_1^2\eta_2^2\eta_3^2\eta_6^2$	4	6	BS + BB : Sections 3 and 5
$\eta_2^4\eta_4^4$	4	8	
η_3^8	4	9	
$\eta_1^4\eta_2^2\eta_4^4$	5	4	BS
$\eta_1^6\eta_3^6$	6	3	BS
η_2^{12}	6	4	BS
$\eta_1^8\eta_2^8$	8	2	BS
η_1^{24}	12	1	BK : Section 2

outlook

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- At higher-loop order the banana integral have interesting relation with Fano variety as classified by [Almkvist, van Straten, Zudilin]