The Scattering Equations in Curved Space

Tim Adamo
DAMTP, University of Cambridge

New Geometric Structures in Scattering Amplitudes

22 September 2014

Work with E. Casali & D. Skinner [arXiv:1409.????]

Motivation

We've learned a lot about perturbative classical GR in recent years:

- Simpler on-shell than Einstein-Hilbert action makes it seem
- Increasingly simple/compact/general formulae for tree-level S-matrix

  [deWitt, Hodges, Cachazo-Geyer, Cachazo-Skinner, Cachazo-He-Yuan, ...]

What are these simple amplitude formulae telling us?

What are these simple amplitude formulae telling us?

There *should* be some simpler formulation of GR

as a *non-linear theory* of gravity!

An analogy...

The Veneziano amplitude:

- Remarkably compact
- Lots of nice properties
- Can be generalized to higher-points

An analogy...

The Veneziano amplitude:

- Remarkably compact
- Lots of nice properties
- Can be generalized to higher-points

But the real upshot is *string theory*!

We have a similar situation with gravity amplitudes:

Remarkably compact/general formulae, but *where* are they coming from?

We have a similar situation with gravity amplitudes:

Remarkably compact/general formulae, but *where* are they coming from?

Partial answer:

- Worldsheet theories which produce these formulae [Skinner, Mason-Skinner, Berkovits, Geyer-Lipstein-Mason]
- Know about *linearized* Einstein equations around flat space

We have a similar situation with gravity amplitudes:

Remarkably compact/general formulae, but *where* are they coming from?

Partial answer:

- Worldsheet theories which produce these formulae [Skinner, Mason-Skinner, Berkovits, Geyer-Lipstein-Mason]
- Know about *linearized* Einstein equations around flat space

Give a formulation of perturbative gravity, linearized around flat space

We want to learn something about the *non-linear* theory!

Back to analogy...

In (closed) string theory, tree-level (sphere) amps:

- Arise from the flat target sigma model
- Give tree-level S-matrix of gravity in $\alpha' \to 0$ limit [Scherk, Yoneya, Scherk-Schwarz]

Back to analogy...

In (closed) string theory, tree-level (sphere) amps:

- Arise from the flat target sigma model
- Give tree-level S-matrix of gravity in $\alpha' \to 0$ limit [Scherk, Yoneya, Scherk-Schwarz]

How to get *non-linear* field equations?

- Formulate non-linear sigma model on curved target space
- Demand worldsheet conformal invariance $\to$ compute β-functions
- Conformal anomaly vanishes as $\alpha' \to 0 \Leftrightarrow$ non-linear field eqns. satisfied

[Callan-Martinec-Perry-Friedan, Banks-Nemeschansky-Sen]

Since non-linear sigma model is an interacting CFT on the worldsheet,

- Must work perturbatively in α'
- Higher powers of α' $\leftrightarrow$ higher-curvature corrections to field equations
  [Gross-Witten, Grisaru-van de Ven-Zanon]

Evident in S-matrix *and* β-function approaches

Since non-linear sigma model is an interacting CFT on the worldsheet,

- Must work perturbatively in α'
- Higher powers of α' $\leftrightarrow$ higher-curvature corrections to field equations
  [Gross-Witten, Grisaru-van de Ven-Zanon]

Evident in S-matrix *and* β-function approaches

But we have a worldsheet theory giving the tree-level S-matrix *EXACTLY*

No higher-derivative corrections

Basic idea

So we want to:

- Formulate the worldsheet theory on a curved target space
- Do it so that the theory is solveable (no background field/perturbative expansion required)
- See non-linear field equations as some sort of anomaly cancellation condition

Starting Point

One particular representation of the tree-level S-matrix [Cachazo-He-Yuan] :

$$\mathcal{M}_{n,0} = \int \frac{1}{\mathrm{vol}\,\mathrm{SL}(2,\mathbb{C})} \frac{|z_1 z_2 z_3|}{\mathrm{d}z_1\,\mathrm{d}z_2\,\mathrm{d}z_3} \prod_{i=4}^{n} \bar{\delta}\left(\sum_{j\neq i} \frac{k_i \cdot k_j}{z_i - z_j}\right) \mathrm{Pf}'(M)\,\mathrm{Pf}'(\widetilde{M})$$

Starting Point

One particular representation of the tree-level S-matrix [Cachazo-He-Yuan] :

$$\mathcal{M}_{n,0} = \int \frac{1}{\mathrm{vol}\,\mathrm{SL}(2,\mathbb{C})} \frac{|z_1 z_2 z_3|}{\mathrm{d}z_1\,\mathrm{d}z_2\,\mathrm{d}z_3} \prod_{i=4}^{n} \bar{\delta}\left(\sum_{j\neq i} \frac{k_i \cdot k_j}{z_i - z_j}\right) \mathrm{Pf}'(M)\,\mathrm{Pf}'(\widetilde{M})$$

$\{z_i\} \subset \Sigma \cong \mathbb{CP}^1$, $\{k_i\}$ null momenta,

$$M = \begin{pmatrix} A & -C^{\mathrm{T}} \\ C & B \end{pmatrix}, \qquad \mathrm{Pf}'(M) = (-1)^{i+j} \frac{\sqrt{\mathrm{d}z_i\,\mathrm{d}z_j}}{z_i - z_j} \mathrm{Pf}(M^{ij}_{ij}),$$

$$A_{ij} = k_i \cdot k_j \frac{\sqrt{\mathrm{d}z_i\,\mathrm{d}z_j}}{z_i - z_j}, \quad B_{ij} = \epsilon_i \cdot \epsilon_j \frac{\sqrt{\mathrm{d}z_i\,\mathrm{d}z_j}}{z_i - z_j}, \quad C_{ij} = \epsilon_i \cdot k_j \frac{\sqrt{\mathrm{d}z_i\,\mathrm{d}z_j}}{z_i - z_j}$$

$A_{ii} = B_{ii} = 0,\ C_{ii} = -\mathrm{d}z_i \sum_{j\neq i} \frac{C_{ij}}{\sqrt{\mathrm{d}z_i\,\mathrm{d}z_j}}$

This representation of $\mathcal{M}_{n,0}$ manifests (gauge)^2=(gravity), and related to BCJ duality

All integrals over $\overline{M}_{0,n}$ fixed by delta functions, imposing the *scattering equations* [Fairlie-Roberts, Gross-Mende, Witten] :

$$i \in \{4, \ldots, n\}, \qquad \sum_{j \neq i} \frac{k_i \cdot k_j}{z_i - z_j} = 0$$

So the locations $\{z_i\} \subset \Sigma$ are *fixed* by the scattering equations.

This representation of $\mathcal{M}_{n,0}$ manifests (gauge)^2=(gravity), and related to BCJ duality

All integrals over $\overline{M}_{0,n}$ fixed by delta functions, imposing the *scattering equations* [Fairlie-Roberts, Gross-Mende, Witten] :

$$i \in \{4, \ldots, n\}, \qquad \sum_{j \neq i} \frac{k_i \cdot k_j}{z_i - z_j} = 0$$

So the locations $\{z_i\} \subset \Sigma$ are *fixed* by the scattering equations.

Structure of $\mathcal{M}_{n,0}$ hints at natural origin...

Worldsheet theory, I

Consider worldsheet action [Mason-Skinner] :

$$S = \frac{1}{2\pi} \int_\Sigma P_\mu \bar{\partial} X^\mu + \Psi_\mu \bar{\partial} \Psi^\mu - \chi P_\mu \Psi^\mu + \widetilde{\Psi}_\mu \bar{\partial} \widetilde{\Psi}^\mu - \widetilde{\chi} P_\mu \widetilde{\Psi}^\mu - \frac{e}{2} P^2$$

$P_\mu \in \Omega^0(\Sigma, K)$ and $\Psi^\mu, \widetilde{\Psi}^\mu \in \Pi\Omega^0(\Sigma, K^{1/2})$

Worldsheet theory, I

Consider worldsheet action [Mason-Skinner] :

$$S = \frac{1}{2\pi}\int_\Sigma P_\mu\,\bar\partial X^\mu + \Psi_\mu\bar\partial\Psi^\mu - \chi P_\mu\Psi^\mu + \widetilde\Psi_\mu\bar\partial\widetilde\Psi^\mu - \widetilde\chi P_\mu\widetilde\Psi^\mu - \frac{e}{2}P^2$$

$P_\mu \in \Omega^0(\Sigma, K)$ and $\Psi^\mu, \widetilde\Psi^\mu \in \Pi\Omega^0(\Sigma, K^{1/2})$

$$\xrightarrow{\text{gauge-fixing}} \frac{1}{2\pi}\int_\Sigma P_\mu\,\bar\partial X^\mu + \Psi_\mu\bar\partial\Psi^\mu + \widetilde\Psi_\mu\bar\partial\widetilde\Psi^\mu \; + S^{\mathrm{gh}}$$

where fixing $e = 0$ enforces the constraint

$$P^2 = 0\,.$$

Scattering equations from the worldsheet

In the presence of vertex operator insertions, P_μ becomes *meromorphic*:

$$\bar{\partial} P_\mu(z) = 2\pi i \, \mathrm{d}z \wedge \mathrm{d}\bar{z} \sum_{i=1}^{n} k_{i\,\mu} \, \delta^2(z - z_i).$$

Scattering equations from the worldsheet

In the presence of vertex operator insertions, P_μ becomes *meromorphic*:

$$\bar{\partial} P_\mu(z) = 2\pi i \, \mathrm{d}z \wedge \mathrm{d}\bar{z} \sum_{i=1}^{n} k_{i\,\mu} \, \delta^2(z - z_i).$$

Likewise, quadratic differential P^2 becomes meromorphic, with residues:

$$\mathrm{Res}_{z=z_i} P^2(z) = k_i \cdot P(z_i) = \mathrm{d}z_i \sum_{j \neq i} \frac{k_i \cdot k_j}{z_i - z_j}$$

Setting $\mathrm{Res}_{z=z_i} P^2(z) = 0$ for $i = 4, \ldots, n$ is sufficient to set $P^2(z) = 0$ globally on Σ.

But these *are* the scattering equations!

$$P^2(z) = 0 \quad \leftrightarrow \quad \mathrm{Res}_{z=z_i} P^2(z) = 0 = \sum_{j \neq i} \frac{k_i \cdot k_j}{z_i - z_j} \quad i \in \{4, \ldots, n\}$$

But these *are* the scattering equations!

$$P^2(z) = 0 \quad \leftrightarrow \quad \mathrm{Res}_{z=z_i} P^2(z) = 0 = \sum_{j \neq i} \frac{k_i \cdot k_j}{z_i - z_j} \quad i \in \{4, \ldots, n\}$$

The condition $P^2(z) = 0$ globally on Σ defines the scattering equations for *any* genus worldsheet [TA-Casali-Skinner]

$$\begin{aligned} g = 0 & \quad (n-3) \times \ \mathrm{Res}_{z=z_i} P^2(z) = 0 \\ g = 1 & \quad (n-1) \times \ \mathrm{Res}_{z=z_i} P^2(z) = 0\,, \quad P^2(z_1) = 0 \\ g \geq 2 & \quad \quad n \times \ \mathrm{Res}_{z=z_i} P^2(z) = 0\,, \quad (3g-3) \times P^2(z_r) = 0 \end{aligned}$$

This theory has a BRST-charge

$$Q = \oint c\, T^{\mathrm{m}} + :bc\partial c: + \frac{\tilde{c}}{2} P^2 + \gamma P_\mu \Psi^\mu + \tilde{\gamma} P_\mu \widetilde{\Psi}^\mu \,,$$

which is nilpotent $Q^2 = 0$ provided the space-time has $d = 10$.

This theory has a BRST-charge

$$Q = \oint c\, T^{\mathrm{m}} + :bc\partial c: + \frac{\tilde{c}}{2} P^2 + \gamma P_\mu \Psi^\mu + \tilde{\gamma} P_\mu \widetilde{\Psi}^\mu \,,$$

which is nilpotent $Q^2 = 0$ provided the space-time has $d = 10$.

Fixed and integrated vertex operators:

$$c\tilde{c}\delta(\gamma)\delta(\tilde{\gamma})\, U \,, \qquad \int_\Sigma \bar{\delta}\left(\mathrm{Res}_z P^2\right)\, V$$

for $U \in \Omega^0(\Sigma, K)$, $V \in \Omega^0(\Sigma, K^2)$.

This theory has a BRST-charge

$$Q = \oint c\, T^{\mathrm{m}} + :bc\partial c: + \frac{\tilde{c}}{2} P^2 + \gamma P_\mu \Psi^\mu + \tilde{\gamma} P_\mu \widetilde{\Psi}^\mu\,,$$

which is nilpotent $Q^2 = 0$ provided the space-time has $d = 10$.

Fixed and integrated vertex operators:

$$c\tilde{c}\delta(\gamma)\delta(\tilde{\gamma})\, U\,, \qquad \int_\Sigma \bar{\delta}\left(\mathrm{Res}_z P^2\right)\, V$$

for $U \in \Omega^0(\Sigma, K)$, $V \in \Omega^0(\Sigma, K^2)$.

Anomalies in BRST-closure $\leftrightarrow$ double contractions between currents

$$P^2\,, \quad P_\mu \Psi^\mu\,, \quad P_\mu \widetilde{\Psi}^\mu\,,$$

and $U\,, V$.

For momentum eigenstates, this constrains:

$$QU = QV = 0 \quad \Leftrightarrow \quad k^2 = 0 = \epsilon \cdot k = \tilde{\epsilon} \cdot k$$

i.e., obey the linearized Einstein equations around flat space

The $g = 0$ correlators in this model reproduce the CHY formulae
[Mason-Skinner]

Other vertex operators for dilatons, B-fields, gravitini, R-R form fields

Explicit amplitude candidates at higher genus passing non-trivial checks
[TA-Casali-Skinner] :

- Modular invariance
- Factorization onto rational functions
- Explicit loop momenta (zero modes of $P_\mu(z)$)

Upshot

So, we have a worldsheet theory that:

- Knows about the entire tree-level S-matrix of type II SUGRA in $d = 10$ *exactly*
- Gives scattering equations in the form $P^2 = 0$
- Enforces the linearized Einstein equations about flat space on vertex operators via BRST-closure

Upshot

So, we have a worldsheet theory that:

- Knows about the entire tree-level S-matrix of type II SUGRA in $d = 10$ *exactly*
- Gives scattering equations in the form $P^2 = 0$
- Enforces the linearized Einstein equations about flat space on vertex operators via BRST-closure

Question: can this theory be extended to an arbitrary curved manifold, with the *non-linear* Einstein equations emerging as an anomaly cancellation condition?

Once more, analogy with strings:

String theory

- Tree-level S-matrix $\xrightarrow{\alpha' \to 0}$ supergravity
- linearized EFEs $\leftrightarrow$ anomalous conformal weights

Worldsheet theory

- *Exact* supergravity tree-level S-matrix
- linearized EFEs $\leftrightarrow$ anomalies w/ currents

Once more, analogy with strings:

String theory

- Tree-level S-matrix $\xrightarrow{\alpha' \to 0}$ supergravity
- linearized EFEs $\leftrightarrow$ anomalous conformal weights

Worldsheet theory

- *Exact* supergravity tree-level S-matrix
- linearized EFEs $\leftrightarrow$ anomalies w/ currents

$\Rightarrow$ Look for *solvable* worldsheet theory with curved target space

Worldsheet theory, II

Naive generalization to curved target, M:

$$S = \frac{1}{2\pi}\int_\Sigma P_\mu \bar{\partial} X^\mu + \bar{\psi}_\mu \bar{\mathrm{D}} \psi^\mu + S^{\mathrm{gh}}$$

$$= \frac{1}{2\pi}\int_\Sigma P_\mu \bar{\partial} X^\mu + \bar{\psi}_\mu \left(\delta^\mu_\nu \bar{\partial} + \Gamma^\mu_{\nu\rho} \bar{\partial} X^\rho\right) \psi^\nu + S^{\mathrm{gh}}$$

with complex fermion $\psi^\mu = \Psi^\mu + i\widetilde{\Psi}^\mu$ to make life easier.

Why this way?

Field redefinition

Make the redefinition

$$\Pi_\mu \equiv P_\mu + \Gamma^\rho_{\mu\nu}\bar{\psi}_\rho\psi^\nu$$

so action becomes:

$$S = \frac{1}{2\pi}\int_\Sigma \Pi_\mu\,\bar{\partial}X^\mu + \bar{\psi}_\mu\,\bar{\partial}\psi^\mu\,.$$

Field redefinition

Make the redefinition

$$\Pi_\mu \equiv P_\mu + \Gamma^\rho_{\mu\nu}\bar{\psi}_\rho\psi^\nu$$

so action becomes:

$$S = \frac{1}{2\pi}\int_\Sigma \Pi_\mu\,\bar{\partial}X^\mu + \bar{\psi}_\mu\,\bar{\partial}\psi^\mu\,.$$

Free action and OPEs:

$$X^\mu(z)\,\Pi_\nu(w) \sim \frac{\delta^\mu_\nu}{z-w}\,, \qquad \psi^\mu(z)\,\bar{\psi}_\nu(w) \sim \frac{\delta^\mu_\nu}{z-w}\,.$$

Field redefinition

Make the redefinition

$$\Pi_\mu \equiv P_\mu + \Gamma^{\rho}_{\mu\nu}\bar{\psi}_\rho \psi^\nu$$

so action becomes:

$$S = \frac{1}{2\pi}\int_\Sigma \Pi_\mu \, \bar{\partial} X^\mu + \bar{\psi}_\mu \, \bar{\partial}\psi^\mu \, .$$

Free action and OPEs:

$$X^\mu(z)\,\Pi_\nu(w) \sim \frac{\delta^\mu_\nu}{z-w}\,, \qquad \psi^\mu(z)\,\bar{\psi}_\nu(w) \sim \frac{\delta^\mu_\nu}{z-w}\,.$$

Covariance non-manifest, due to transformation:

$$\tilde{\Pi}_\mu = \frac{\partial X^\nu}{\partial \tilde{X}^\mu}\Pi_\nu + \frac{\partial^2 X^\kappa}{\partial \tilde{X}^\mu \partial \tilde{X}^\nu}\frac{\partial \tilde{X}^\nu}{\partial X^\sigma}\bar{\psi}_\kappa \psi^\sigma$$

Classical currents

Action has fermionic symmetries generated by:

$$\mathcal{G}^{\circ} = \psi^{\mu}\Pi_{\mu}\,, \qquad \bar{\mathcal{G}}^{\circ} = g^{\mu\nu}\bar{\psi}_{\mu}\left(\Pi_{\nu} - \Gamma^{\rho}_{\nu\sigma}\bar{\psi}_{\rho}\psi^{\sigma}\right)\,.$$

Classical currents

Action has fermionic symmetries generated by:

$$\mathcal{G}^{\circ} = \psi^{\mu}\Pi_{\mu}\,, \qquad \bar{\mathcal{G}}^{\circ} = g^{\mu\nu}\bar{\psi}_{\mu}\left(\Pi_{\nu} - \Gamma^{\rho}_{\nu\sigma}\bar{\psi}_{\rho}\psi^{\sigma}\right)\,.$$

Classically, obey the algebra $\{\mathcal{G}^{\circ},\mathcal{G}^{\circ}\} = \{\bar{\mathcal{G}}^{\circ},\bar{\mathcal{G}}^{\circ}\} = 0\,, \{\mathcal{G}^{\circ}\,,\bar{\mathcal{G}}^{\circ}\} = \mathcal{H}^{\circ}$ with

$$\mathcal{H}^{\circ} = g^{\mu\nu}\left(\Pi_{\mu} - \Gamma^{\rho}_{\mu\sigma}\bar{\psi}_{\rho}\psi^{\sigma}\right)\left(\Pi_{\nu} - \Gamma^{\kappa}_{\nu\lambda}\bar{\psi}_{\kappa}\psi^{\lambda}\right) - \frac{1}{2}\psi^{\mu}\psi^{\nu}\bar{\psi}_{\rho}\bar{\psi}_{\sigma}R^{\rho\sigma}{}_{\mu\nu}$$

Classical currents

Action has fermionic symmetries generated by:

$$\mathcal{G}^\circ = \psi^\mu \Pi_\mu \,, \qquad \bar{\mathcal{G}}^\circ = g^{\mu\nu} \bar{\psi}_\mu \left(\Pi_\nu - \Gamma^\rho_{\nu\sigma} \bar{\psi}_\rho \psi^\sigma \right) \,.$$

Classically, obey the algebra $\{\mathcal{G}^\circ, \mathcal{G}^\circ\} = \{\bar{\mathcal{G}}^\circ, \bar{\mathcal{G}}^\circ\} = 0 \,, \{\mathcal{G}^\circ \,, \bar{\mathcal{G}}^\circ\} = \mathcal{H}^\circ$ with

$$\mathcal{H}^\circ = g^{\mu\nu} \left(\Pi_\mu - \Gamma^\rho_{\mu\sigma} \bar{\psi}_\rho \psi^\sigma \right) \left(\Pi_\nu - \Gamma^\kappa_{\nu\lambda} \bar{\psi}_\kappa \psi^\lambda \right) - \frac{1}{2} \psi^\mu \psi^\nu \bar{\psi}_\rho \bar{\psi}_\sigma R^{\rho\sigma}{}_{\mu\nu}$$

These are analogues of the flat space currents:

$$\psi^\mu P_\mu \to \mathcal{G} \,, \quad g^{\mu\nu} \bar{\psi}_\mu P_\nu \to \bar{\mathcal{G}} \,, \quad P^2 \to \mathcal{H}$$

BRST charge

Gauge these currents $\Rightarrow$

$$Q = \oint c\, T^{\mathrm{m}} + :bc\partial c: + \frac{\tilde{c}}{2}\,\mathcal{H}^{\circ} + \bar{\gamma}\,\mathcal{G}^{\circ} + \gamma\,\bar{\mathcal{G}}^{\circ}$$

Does this agree with what we're expecting?

At the naive level, yes:

- Free OPEs
- Only conformal anomaly condition remains $d = 10$

At the naive level, yes:

- Free OPEs
- Only conformal anomaly condition remains $d = 10$

So where are potential anomalies?

BRST-charge is nilpotent iff

$$\mathcal{G}^{\circ}(z)\,\mathcal{G}^{\circ}(w) \sim 0 \sim \bar{\mathcal{G}}^{\circ}(z)\,\bar{\mathcal{G}}^{\circ}(w)\,, \qquad \mathcal{G}^{\circ}(z)\,\bar{\mathcal{G}}^{\circ}(w) \sim \frac{\mathcal{H}^{\circ}}{z-w}\,.$$

But we only know this *classically*; need to extend to quantum level

Quantum issues

Before we can look at these anomalies, we still have lots to worry about at the quantum level:

- Diffeomorphism covariance of the fields
- Diffeomorphism covariance of the currents

In other words, *do the currents even make sense quantum mechanically*?

Infinitesimal diffeomorphism on M generated by vector field $V = V^{\mu}\partial_{\mu}$.

At quantum level, look for an operator $\mathcal{O}_V$ obeying:

$$\mathcal{O}_V(z)\,\mathcal{O}_W(w) \sim \frac{\mathcal{O}_{[V,W]}(w)}{z-w}$$

and acting on fields as:

$$\mathcal{O}_V(z)\,X^{\mu}(w) \sim \frac{V^{\mu}}{z-w}\,, \qquad \mathcal{O}_V(z)\,\psi^{\mu}(w) \sim \frac{\partial_{\nu}V^{\mu}\,\psi^{\nu}}{z-w}\,,$$

$$\mathcal{O}_V(z)\,\bar{\psi}_{\mu}(w) \sim -\frac{\partial_{\mu}V^{\nu}\,\bar{\psi}_{\nu}}{z-w}\,,$$

$$\mathcal{O}_v(z)\,\Pi_{\mu}(w) \sim -\frac{\partial_{\mu}V^{\nu}\,\Pi_{\nu} + \partial_{\mu}\partial_{\nu}V^{\rho}\,\bar{\psi}_{\rho}\psi^{\nu}}{z-w}$$

Infinitesimal diffeomorphism on M generated by vector field $V = V^{\mu}\partial_{\mu}$.

At quantum level, look for an operator $\mathcal{O}_V$ obeying:

$$\mathcal{O}_V(z)\, \mathcal{O}_W(w) \sim \frac{\mathcal{O}_{[V,W]}(w)}{z-w}$$

and acting on fields as:

$$\mathcal{O}_V(z)\, X^{\mu}(w) \sim \frac{V^{\mu}}{z-w}\,, \qquad \mathcal{O}_V(z)\, \psi^{\mu}(w) \sim \frac{\partial_{\nu} V^{\mu}\, \psi^{\nu}}{z-w}\,,$$

$$\mathcal{O}_V(z)\, \bar{\psi}_{\mu}(w) \sim -\frac{\partial_{\mu} V^{\nu}\, \bar{\psi}_{\nu}}{z-w}\,,$$

$$\mathcal{O}_v(z)\, \Pi_{\mu}(w) \sim -\frac{\partial_{\mu} V^{\nu}\, \Pi_{\nu} + \partial_{\mu}\partial_{\nu} V^{\rho}\, \bar{\psi}_{\rho}\psi^{\nu}}{z-w}$$

Implemented by:

$$\mathcal{O}_V = -\left(V^{\mu}\Pi_{\mu} + \partial_{\nu} V^{\mu}\bar{\psi}_{\mu}\psi^{\nu}\right)$$

Quantum currents

How does $\mathcal{O}_V$ act on composite operators like $\mathcal{G}^{\circ}, \bar{\mathcal{G}}^{\circ}$?

Quantum currents

How does $\mathcal{O}_V$ act on composite operators like $\mathcal{G}^\circ, \bar{\mathcal{G}}^\circ$?

On any $J(\mathcal{F}(X))$, infinitesimal diffeos should act geometrically:

$$\mathcal{O}_V(z)\, J(\mathcal{F}(X))(w) \sim \cdots + \frac{J(\mathcal{L}_V\mathcal{F})}{z-w} + \cdots$$

But our currents $\mathcal{G}^\circ, \bar{\mathcal{G}}^\circ$ don't obey this. (double contractions!)

Quantum currents

How does $\mathcal{O}_V$ act on composite operators like $\mathcal{G}^\circ, \bar{\mathcal{G}}^\circ$?

On any $J(\mathcal{F}(X))$, infinitesimal diffeos should act geometrically:

$$\mathcal{O}_V(z)\, J(\mathcal{F}(X))(w) \sim \cdots + \frac{J(\mathcal{L}_V \mathcal{F})}{z-w} + \cdots$$

But our currents $\mathcal{G}^\circ, \bar{\mathcal{G}}^\circ$ don't obey this. (double contractions!)

Solution: add *quantum* corrections

To fix OPE with $\mathcal{O}_V$, take

$$\mathcal{G} = \mathcal{G}^\circ + \partial\left(\psi^\mu \Gamma^\nu_{\mu\nu}\right)$$

$$\bar{\mathcal{G}} = \bar{\mathcal{G}}^\circ - g^{\nu\sigma}\partial\left(\bar{\psi}_\mu \Gamma^\mu_{\nu\sigma}\right)$$

Great, but now $\mathcal{G}$, $\bar{\mathcal{G}}$ no longer worldsheet primaries.

Resolution $\Rightarrow$ quantum correction to stress tensor:

$$T = -\Pi_\mu \, \partial X^\mu - \frac{1}{2}\bar{\psi}_\mu \, \partial \psi^\mu - \frac{1}{2}\psi^\mu \, \partial \bar{\psi}_\mu - \frac{1}{2}\partial^2 \log(\sqrt{g})$$

Note: doesn't alter central charge!

Action now invariant under quantum charges, and free OPEs unaffected

Some observations:

- Similar methods for removing anomalous OPEs in study of *curved $\beta\gamma$-systems* [Nekrasov, Witten]
- See also math literature, *sheaves of chiral algebras, chiral de Rham complex* [Malikov-Schechtman-Vaintrob, Gorbounov-Malikov-Schechtman, Ben-Zvi-Heluani-Szczesny, Frenkel-Losev-Nekrasov, Ekstrand-Heluani-Kallen-Zabzine]
- Related constructions in 1^{st}-order formalism for string theory [Schwarz-Tseytlin, Losev-Marshakov-Zeitlin]

Quantum model

We now have a well-defined worldsheet theory, and a BRST operator built from ghosts and the currents:

Quantum Currents

$$\begin{aligned}
\mathcal{G} &= \psi^{\mu}\Pi_{\mu} + \partial\left(\psi^{\mu}\Gamma^{\nu}_{\mu\nu}\right) \\
\bar{\mathcal{G}} &= g^{\mu\nu}\bar{\psi}_{\mu}\left(\Pi_{\nu} - \Gamma^{\rho}_{\nu\sigma}\bar{\psi}_{\rho}\psi^{\sigma}\right) - g^{\nu\sigma}\partial\left(\bar{\psi}_{\mu}\Gamma^{\mu}_{\nu\sigma}\right) \\
T &= -\Pi_{\mu}\,\partial X^{\mu} - \frac{1}{2}\bar{\psi}_{\mu}\,\partial\psi^{\mu} - \frac{1}{2}\psi^{\mu}\,\partial\bar{\psi}_{\mu} - \frac{1}{2}\partial^{2}\log(\sqrt{g})
\end{aligned}$$

Only potential anomalies to $Q^2 = 0$ from algebra of currents

$$\mathcal{G}(z)\,\mathcal{G}(w) \sim 0 \sim \bar{\mathcal{G}}(z)\,\bar{\mathcal{G}}(w)\,, \qquad \mathcal{G}(z)\,\bar{\mathcal{G}}(w) \sim \frac{\mathcal{H}}{z-w}$$

Anomaly calculation

Do the OPEs (lots of fun!) and find:

$$\mathcal{G}(z)\,\mathcal{G}(w) \sim 0\,,$$

$$\bar{\mathcal{G}}(z)\,\bar{\mathcal{G}}(w) \sim \frac{1}{2}\frac{\bar{\psi}_\mu\bar{\psi}_\nu\bar{\psi}_\rho\psi^\sigma}{z-w}R_\sigma{}^{\mu\nu\rho} + \frac{\partial\left(\bar{\psi}_\mu\bar{\psi}_\nu R^{\mu\nu}\right)}{z-w} + 2\frac{\bar{\psi}_\mu\bar{\psi}_\nu\partial X^\sigma}{z-w}\left[\Gamma^\nu_{\alpha\beta}R^{\beta\alpha\mu}{}_\sigma + \Gamma^\alpha_{\sigma\beta}(R^{\mu\beta\nu}{}_\alpha + R^{\nu\beta\mu}{}_\alpha)\right]$$

$$\mathcal{G}(z)\,\bar{\mathcal{G}}(w) \sim \frac{2}{(z-w)^3}R + 2\frac{(\Gamma^\mu_{\sigma\nu}\partial X^\sigma + \psi^\mu\bar{\psi}_\nu)}{(z-w)^2}R^\mu_\nu + \frac{\mathcal{H}}{z-w}$$

The only anomaly cancellation conditions are:

$$R_{\mu\nu} = 0 = R\,,$$

the vacuum Einstein equations!

The only anomaly cancellation conditions are:

$$R_{\mu\nu} = 0 = R\,,$$

the vacuum Einstein equations!

Note:

- Free OPEs, so anomalies are *exact*
- No background field expansion
- No perturbative (α') expansion on worldsheet

Other fields

Can also add dilaton and B-field:

$$
\begin{aligned}
\mathcal{G} &= \psi^\mu \Pi_\mu + \frac{1}{6} H_{\mu\nu\rho}\psi^\mu\psi^\nu\psi^\rho + \partial\left(\psi^\mu \Gamma^\nu_{\mu\nu}\right) - 2\partial\left(\psi^\mu\partial_\mu\Phi\right) \\
\bar{\mathcal{G}} &= g^{\mu\nu}\bar{\psi}_\mu\left(\Pi_\nu - \Gamma^\rho_{\nu\sigma}\bar{\psi}_\rho\psi^\sigma\right) + \frac{1}{6}H^{\mu\nu\rho}\bar{\psi}_\mu\bar{\psi}_\nu\bar{\psi}_\rho \\
&\quad - g^{\nu\sigma}\partial\left(\bar{\psi}_\mu\Gamma^\mu_{\nu\sigma}\right) - 2\partial\left(\bar{\psi}_\mu g^{\mu\nu}\partial_\nu\Phi\right) \\
T &= -\Pi_\mu\,\partial X^\mu - \frac{1}{2}\bar{\psi}_\mu\,\partial\psi^\mu - \frac{1}{2}\psi^\mu\,\partial\bar{\psi}_\mu - \frac{1}{2}\partial^2\log\left(\sqrt{g}\,\mathrm{e}^{-2\Phi}\right)
\end{aligned}
$$

and do the same sort of calculations...

The only anomaly cancellation conditions are:

Field Equations

$$
\begin{aligned}
R_{\mu\nu} - \frac{1}{4} H_{\mu\rho\sigma} H_{\nu}^{\rho\sigma} + 2\nabla_\mu \nabla_\nu \Phi &= 0\,, \\
\nabla_\rho H^{\rho}_{\mu\nu} - 2 H^{\rho}_{\mu\nu} \nabla_\rho \Phi &= 0\,, \\
R + 4\nabla_\mu \nabla^\mu \Phi - 4\nabla_\mu \Phi\, \nabla^\mu \Phi - \frac{H^2}{12} &= 0\,.
\end{aligned}
$$

Back to scattering equations

In flat space, the scattering equations were $P^2 = 0$.

On M, they become

$$\mathcal{G}(z)\,\bar{\mathcal{G}}(w) \sim 0.$$

Back to scattering equations

In flat space, the scattering equations were $P^2 = 0$.

On M, they become

$$\mathcal{G}(z)\,\bar{\mathcal{G}}(w) \sim 0.$$

This has a *quasi-classical* piece, $\mathcal{H} = 0$, and *quantum* pieces.

The quantum pieces of the scattering equations in curved space *are* the field equations!

Summary

Worldsheet CFT which is

- Solvable (basically free)
- Background independent
- Encodes scattering equations and field equations
- Reduces to flat space model (linearize $\mathcal{H}$ around flat space to get V)