Tree-level scattering amplitudes in $\mathcal{N} = 4$ SYM from integrability

Tomek Łukowski

Mathematical Institute, University of Oxford

New Geometric Structures in Scattering Amplitudes

University of Oxford

23.09.2014

Based on:

L. Ferro, TŁ, C. Meneghelli, J. Plefka, M. Staudacher – 1212.0850

L. Ferro, TŁ, C. Meneghelli, J. Plefka, M. Staudacher – 1308.3494

N. Kanning, TŁ, M. Staudacher – 1403.3382

L. Ferro, TŁ, M. Staudacher – 1407.6736

Introduction

- **Main focus**: Understand and use integrable structures present in four-dimensional quantum field theories.
- Quantum integrability – concept originating from 1+1 dimensional quantum systems.
  $\rightarrow$ Existence of an infinite dimensional symmetry.
- Integrability in 1+3 dimensions: integrable structures come from some dual two-dimensional description.
- Focus on the planar limit of maximally supersymmetric Yang-Mills theory ($\mathcal{N} = 4$ SYM) in four dimensions:
  - scaling dimensions $\leftrightarrow$ energies of worldsheet excitations [many authors, 2003-]
  - polygonal Wilson loops $\leftrightarrow$ GKP string excitations [Benjamin's and Pedro's talks]
  - scattering amplitudes at strong coupling $\leftrightarrow$ minimal surfaces [Alday, Maldacena, Sever, Vieira]
  - scattering amplitudes at weak coupling $\leftrightarrow$ inhomogeneous spin chains [this talk]

Our motivation

- Integrability proved its usefulness in finding all-loop and finite coupling results for scaling dimensions of gauge invariant operators. We hope the history will repeat itself for scattering amplitudes.
- We aim in constraining or constructing scattering amplitudes using powerful tools of integrable models, e. g. **quantum inverse scattering method** (QISM).
- Amplitudes suffer from infrared divergencies. Most popular method to regulate – dimensional regularization. Away from four dimensions large part of the nice structure disappears. Spectral parameters promise a new way of regulating divergencies while staying in four dimensions!

Amplitudes in $\mathcal{N} = 4$ SYM

We consider color-ordered scattering amplitudes of superfields

$$\Phi = G^{+} + \tilde{\eta}^{A}\Gamma_{A} + \frac{1}{2!}\tilde{\eta}^{A}\tilde{\eta}^{B}S_{AB} + \frac{1}{3!}\tilde{\eta}^{A}\tilde{\eta}^{B}\tilde{\eta}^{C}\epsilon_{ABCD}\,\overline{\Gamma}^{D} + \frac{1}{4!}\tilde{\eta}^{A}\tilde{\eta}^{B}\tilde{\eta}^{C}\tilde{\eta}^{D}\epsilon_{ABCD}\,G^{-}$$

The amplitudes $\mathcal{A}_{n,k}$ are labeled by two numbers:

- number of particles – n
- MHV level – $\tilde{\eta}^{4k}$, $\quad k = 2, \ldots n-2\,,$

$$\mathcal{A}_{n} = \mathcal{A}_{n,2} + \tilde{\eta}^{4}\,\mathcal{A}_{n,3} + \ldots + \tilde{\eta}^{4k-8}\,\mathcal{A}_{n,k-2}$$

All particles are massless: $p^{2} = 0 \Rightarrow p^{\alpha\dot{\alpha}} = \lambda^{\alpha}\tilde{\lambda}^{\dot{\alpha}}$.

On-shell superspace – $\Lambda^{\mathcal{A}} = (\lambda^{\alpha}, \tilde{\lambda}^{\dot{\alpha}}, \tilde{\eta}^{A})$

Parke-Taylor formula for MHV amplitudes : [Parke, Taylor]

$$\mathcal{A}_{n,2} = \frac{\delta^{4}(P)\delta^{8}(Q)}{\langle 12\rangle\langle 23\rangle\ldots\langle n1\rangle}\,, \qquad \langle ij\rangle = \epsilon_{\alpha\beta}\lambda_{i}^{\alpha}\lambda_{j}^{\beta}$$

Twistors - natural coordinates to describe scattering amplitudes

Twistor variables: $\mathcal{W}^{\mathcal{A}} = (\tilde{\mu}^{\alpha}, \tilde{\lambda}^{\dot{\alpha}}, \tilde{\eta}^{A})$ [Penrose]

where $\tilde{\mu}$ is the Fourier transform of λ.

- Conformal symmetry [Witten]

$$\sum_i \mathcal{W}_i^{\mathcal{A}} \frac{\partial}{\partial \mathcal{W}_i^{\mathcal{B}}} \mathcal{A}_{n,k} = 0$$

Momentum twistors: $\mathcal{Z}^{\mathcal{A}} = (\lambda^{\alpha}, \mu^{\dot{\alpha}}, \eta^{A})$ [Hodges]

- Dual conformal symmetry [Drummond, Henn, Korchemsky, Sokatchev], [Drummond,Ferro]

$$\sum_i \mathcal{Z}_i^{\mathcal{A}} \frac{\partial}{\partial \mathcal{Z}_i^{\mathcal{B}}} \frac{\mathcal{A}_{n,k}}{\mathcal{A}_{n,2}} = 0$$

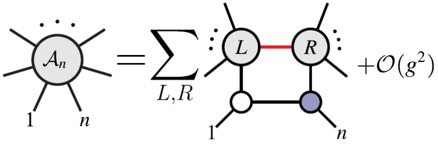
Yangian algebra generators in twistor space [Drummond, Henn, Plefka]

$$J^{\mathcal{A}\mathcal{B}} = \sum_i \mathcal{W}_i^{\mathcal{A}} \frac{\partial}{\partial \mathcal{W}_i^{\mathcal{B}}}, \qquad \hat{J}^{\mathcal{A}\mathcal{B}} = \sum_{i<j} \left(\mathcal{W}_i^{\mathcal{A}} \frac{\partial}{\partial \mathcal{W}_i^{\mathcal{C}}} \mathcal{W}_j^{\mathcal{C}} \frac{\partial}{\partial \mathcal{W}_j^{\mathcal{B}}} - (i \leftrightarrow j) \right) + \sum_i v_i \mathcal{W}_i^{\mathcal{A}} \frac{\partial}{\partial \mathcal{W}_i^{\mathcal{B}}}$$

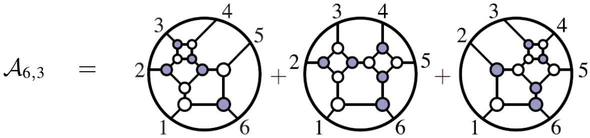
Analogous expressions for momentum twistors. v_i – evaluation representation parameters.

BCFW recursion relation for scattering amplitudes in $\mathcal{N} = 4$ SYM

- BCFW recursion relation (based on the residue theorem): [Arkani-Hamed, Bourjaily,Cachazo, Caron-Huot, Trnka]



- Example solution to the tree-level BCFW recursion relation



- One can associate a **permutation** to each on-shell diagram.
- One can associate an integral over an auxiliary real/complex Graßmannian to each such diagram. All such integrals are **Yangian invariant** for suitable integration contours.
- Real Graßmannians – on-shell diagrams correspond to cells of positive Graßmannian. [Arkani-Hamed, Bourjaily, Cachazo, Goncharov, Postnikov, Trnka]
- Complex Graßmannians – on-shell diagrams related to residues of Graßmannian integrals

From amplitudes to spin chains

We consider planar theory $\to$ color ordered amplitudes

scattering amplitude in $\mathcal{N}=4$ SYM	$\leftrightarrow$	$(\mathfrak{p})\mathfrak{su}(2,2\|4)$ spin chain
particle	$\leftrightarrow$	spin chain site
number of particles n	$\leftrightarrow$	length of spin chain
MHV degree k	$\leftrightarrow$	?

- spin chain state is a polynomial/function in oscillators $\bar{\mathbf{a}}_i^{\alpha}, \bar{\mathbf{b}}_i^{\dot{\alpha}}, \bar{\mathbf{c}}_i^{A}$ acting on the Fock vacuum and constraint by ($\mathfrak{c}_i$ – central charge of $\mathfrak{su}(2,2|4)$)

$$2 + n_i^{\mathbf{a}} - n_i^{\mathbf{b}} - n_i^{\mathbf{c}} = \mathfrak{c}_i$$

- amplitude is a function/distribution of $\lambda_i^{\alpha}, \tilde{\lambda}_i^{\dot{\alpha}}, \tilde{\eta}_i^{A}$ with the constrained (h_i – superhelicity)

$$\left(2 + \lambda_i \frac{\partial}{\partial \lambda_i} - \tilde{\lambda}_i \frac{\partial}{\partial \tilde{\lambda}_i} - \tilde{\eta}_i \frac{\partial}{\partial \tilde{\eta}_i}\right) \mathcal{A} = 2(1 - h_i)\mathcal{A}$$

Task: use QISM to construct Yangian invariants of the **inhomogeneous** $\mathfrak{gl}(N|M)$ spin chain

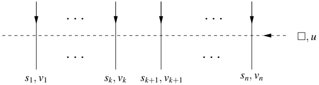
Yangian invariance = monodromy eigenproblem

- Alternative way of defining Yangian invariance for inhomogeneous spin chains

$$M^{AB}(u)|\Psi\rangle = \delta^{AB}|\Psi\rangle\,. \qquad (\star)$$

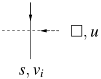
The **monodromy matrix** is defined as

$$M(u) = L_1(u,v_1)\dots L_n(u,v_n) =$$



with the **Lax operators**

$$L_i(u,v_i) = N(u,v_i)\left((u-v_i) + \sum_{A,B} e_{AB}\,J_i^{AB}\right) =$$



- Expanding the monodromy matrix around $u \to \infty$ we find

$$M^{AB}(u) = \delta^{AB} + \frac{1}{u}J^{AB} + \frac{1}{u^2}\hat{J}^{AB} + \dots$$

- Monodromy eigenproblem is equivalent to demanding Yangian invariance: $|\Psi\rangle$ is annihilated by all Yangian generators!

- Solution to $(\star)$ can be found using the Algebraic Bethe Ansatz. Focus on highest weight representations of $\mathfrak{su}(2)$ and define

$$M(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}$$

- The monodromy eigenproblem is equivalent to the conditions:

$$A(u)|\Psi\rangle = D(u)|\Psi\rangle = |\Psi\rangle \qquad\qquad B(u)|\Psi\rangle = C(u)|\Psi\rangle = 0$$

- Two oscillator realizations of the algebra (**symmetric** and **dual** realizations)

$$J^{AB} = \bar{\mathbf{a}}^A \mathbf{a}^B \,, \qquad\qquad \bar{J}^{AB} = -\bar{\mathbf{b}}^B \mathbf{b}^A$$

- Consider a particular (inhomogeneous) quantum space

$$\bar{V}_{s_1} \otimes \ldots \bar{V}_{s_k} \otimes V_{s_{k+1}} \otimes V_{s_n}$$

- Construct a reference state, which is highest weight, that is $C(u)|\Omega\rangle = 0$

$$|\Omega\rangle = \omega_1 \otimes \ldots \otimes \omega_n \,, \qquad \omega_i = \begin{cases} (\bar{\mathbf{b}}_i^2)^{s_i}|\bar{0}\rangle & \text{for } i = 1, \ldots, k \\ (\bar{\mathbf{a}}_i^1)^{s_i}|0\rangle & \text{for } i = k+1, \ldots, n \end{cases}$$

and make a Bethe ansatz for the Yangian invariant in the form

$$|\Psi\rangle = B(u_1)\ldots B(u_F)|\Omega\rangle$$

- It is Yangian invariant if and only if the Bethe equations are satisfied

$$\frac{Q(u)}{Q(u+1)} = \prod_{i=1}^{k} \frac{u - v_i - s_i - 1}{u - v_i - 1} \,,$$

$$\prod_{i=1}^{k} \frac{u - v_i - s_i - 2}{u - v_i - 2} \prod_{i=k+1}^{n} \frac{u - v_i + s_i}{u - v_i} = 1$$

with the Baxter polynomial $Q(u) = \prod_{i=1}^{F}(u - u_i)$.

- From Bethe equations to permutations ($v_i^+ = v_i \pm \frac{s_i}{2} + 2$, $v_i^- = v_i \mp \frac{s_i}{2}$):

$$\prod_{i=1}^{n}(u - v_i^+) = \prod_{i=1}^{n}(u - v_i^-)$$

All solutions are of the form $v_{\sigma(i)}^+ = v_i^-$ for some **permutation** σ!

- Sample invariants:

$$|\Psi\rangle_{2,1} = (\bar{\mathbf{b}}_1 \cdot \bar{\mathbf{a}}_2)^{s_2}|\mathbf{0}\rangle \qquad \sigma_{2,1} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$$

$$|\Psi\rangle_{3,1} = (\bar{\mathbf{b}}_1 \cdot \bar{\mathbf{a}}_2)^{s_2}(\bar{\mathbf{b}}_1 \cdot \bar{\mathbf{a}}_3)^{s_3}|\mathbf{0}\rangle \qquad \sigma_{3,1} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$$

$$|\Psi\rangle_{3,2} = (\bar{\mathbf{b}}_1 \cdot \bar{\mathbf{a}}_3)^{s_1}(\bar{\mathbf{b}}_2 \cdot \bar{\mathbf{a}}_3)^{s_2}|\mathbf{0}\rangle \qquad \sigma_{3,2} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$$

with the Fock vacuum

$$|\mathbf{0}\rangle = |\bar{0}\rangle \otimes \ldots \otimes |\bar{0}\rangle \otimes |0\rangle \otimes \ldots \otimes |0\rangle$$

From Yangian invariants to Graßmannian integrals

- We represent harmonic oscillators as

$$\bar{\mathbf{a}}_i^A, \mathbf{b}_i^A \leftrightarrow \mathcal{W}_i^A \qquad \mathbf{a}_i^A, \bar{\mathbf{b}}_i^A = \frac{\partial}{\partial \mathcal{W}_i^A}$$

- Building blocks for invariants

$$\mathcal{B}_{ij}(u) = \left(\mathcal{W}_i \cdot \frac{\partial}{\partial \mathcal{W}_j}\right)^u$$

and the Fock vacuum

$$|\mathbf{0}\rangle = \prod_{i=1}^{k} \delta^{4|4}(\mathcal{W}_i)$$

- Using the **integral representation** of $\mathcal{B}$-operators

$$(\mathcal{W}_i \cdot \partial_{\mathcal{W}_j})^u = \int \frac{d\alpha}{\alpha^{1+u}} e^{\alpha \mathcal{W}_i \cdot \partial_{\mathcal{W}_j}}$$

one obtains, after change of variables, the integral over the Graßmannian space $G(2,4)$:

$$|\Psi\rangle_{4,2} = \int \frac{d^{2\times 2}C}{(12)^{1+v_4^- - v_1^-}(23)^{1+v_1^- - v_2^-}(34)^{1+v_2^- - v_3^-}(41)^{1+v_3^- - v_4^-}} \delta^{4|4}(C\cdot\mathcal{W})$$

where $\quad C = \begin{pmatrix} 1 & 0 & c_{13} & c_{14} \\ 0 & 1 & c_{23} & c_{24} \end{pmatrix} \quad$ and $\quad (ij) = c_{1i}c_{2j} - c_{2i}c_{1j}$

Deformations and BCFW recursion

- **So far**: we derived deformed Graßmannian integrals associated to Yangian invariants with non-zero evaluation parameters v_i. Each such integral can be associated a deformed on-shell diagram. Inhomogeneities v_i are indispensable for the integrability-based construction to work.
- The non-deformed amplitude is a sum of BCFW terms.
- From the QISM point of view each BCFW term can be deformed, however, the eigenproblems for various invariants differ

$$M_\sigma(u,\{v_i\})|\Psi\rangle_\sigma = |\Psi\rangle_\sigma\,, \qquad v^+_{\sigma(i)} = v^-_i$$

- For non-zero evaluation parameters the sum of Yangian invariants is not Yangian invariant $\rightarrow$ we cannot add deformed on-shell diagrams

How can we define deformed amplitudes?

Top cell and BCFW recursion relation

- Distinguished role of permutations given by **shifts**

$$\sigma_{n,k}(i) = i + k \,(\mathrm{mod}\; n)$$

They correspond to the so-called top cells of the positive Graßmannian $G(k, n)$.

Graßmannian integral for top cell [Arkani-Hamed, Cachazo, Cheung, Kaplan]

$$\int \frac{d^{k\cdot n}C}{\mathrm{vol}(\mathrm{GL}(k))} \frac{\delta^{4k|4k}(C \cdot \mathcal{W})}{(1, \ldots, k)(2, \ldots, k+1) \ldots (n, \ldots, n+k-1)}$$

- Graßmannian integrals associated with any other permutation can be obtained by evaluating a proper residue of the integral for top cell.
- The BCFW recursion relation can be equivalently written as a proper choice of integration contour in the above integral

  → The top cell integral „*knows everything*" about the amplitude; BCFW recursion allows to extract this knowledge.

Meromorphicity lost and gained

[Ferro, TŁ, Staudacher]

Deformed Graßmannian contour integral for top cell

$$\int \frac{d^{k\cdot n}C}{\mathrm{vol(GL(k))}} \frac{\delta^{4k|4k}(C\cdot\mathcal{W})}{(1,\,\dots,k)^{1+v_k^+-v_1^-}\dots(n,\,\dots,n+k-1)^{1+v_{k-1}^+-v_n^-}}\,.$$

see also [Bargheer, Huang, Loebbert, Yamazaki]

- Choosing the parameters $v_j^\pm$ to be non-integer, we see that the poles in the variables c_{aj} generically turn into branch points.
- **Important point**: We can no longer use the BCFW recursion relations, as they are based on the residue theorem, which does not apply anymore.
- What we can hope to gain is complete meromorphicity in suitable combinations of the deformation parameters $v_j^\pm$. Our ultimate hope is that this will fix the contours uniquely.

Amplitudes as solutions to differential equation

[Ferro, TŁ, Staudacher]

- Yangian invariance as a **differential equation**

$$L_1(u, v_1) \dots L_n(u, v_n)|\Psi\rangle = |\Psi\rangle \,, \qquad L_i(u, v_i) = \left(u - v_i + \mathcal{W}_i \frac{\partial}{\partial \mathcal{W}_i}\right)$$

  Second order differential equation in many variables → many independent solutions.

- Example ($v_i = 0, n = 6, k = 3$)

$$\int_\Gamma d\tau \, \frac{P(\tau, \eta)}{\tau(1-\tau)(1-z_1\tau)(1-z_2\tau)(1-z_3\tau)}$$

  where $P(\tau, \eta)$ is a polynomial in τ and fermionic variables η, and z_i are known function of external twistors.

  This integral is Yangian invariant if we take Γ to be a **closed contour**. There are five independent closed contours → circles around the poles. The amplitude is a combination of residues evaluated at these poles.

- For $v_i \neq 0$: poles turns into branch points. One needs to look for a different family of closed contours → Pochhammer contours.

A first look at deformed $\mathcal{A}_{6,3}$

[Ferro, TŁ, Staudacher]

- Deformed Graßmannian integral for the top cell of $G(3,6)$ reduces to the following one-dimensional integral

$$\int d\tau\, \tau^{\alpha_6 - 1}(1-\tau)^{\alpha_5 - 1}\prod_{i=2}^{4}(1 - z_i\, \tau)^{\alpha_i - 1}P(\tau,\eta)$$

  where α_i are known combinations of v_i.

  $\rightarrow$ This integral is of the **Lauricella F_D hypergeometric type**.

- We want to find a proper combination of solutions, which after taking the limit $v_i \rightarrow 0$ reduces to the expression for amplitude – this combination should be given by the **deformed version of the BCFW recursion relation** – still waiting to be discovered.

Other approaches to scattering amplitudes in $\mathcal{N} = 4$ SYM

- Over the years many different expressions for tree-level amplitudes in $\mathcal{N} = 4$ SYM were written down:
  - MHV vertex formalism [Cachazo, Svrcek, Witten]
  - scattering equations in four dimensions [Cachazo, He, Yuan]
  - ambitwistor strings in four dimensions [Geyer, Lipstein, Mason]
  - amplituhedron [Arkani-Hamed, Trnka]
- Non-trivial to check their Yangian invariance. What is the meaning of the deformation parameters?
- Can the construction of spectral parameter deformations, when written in different framework, resolve the problems we encountered in the Graßmannian integrals approach? It might be easier to generalize our construction to the loop-level using different formalism!

Outlook

- Work out general deformed tree-level amplitudes explicitly.
- Write BCFW recursion relations for deformed amplitudes.
- Explore exciting relations to generalized multi-variate hypergeometric functions.
- Investigate the relation to positivity – relation to amplituhedron?
- Establish that the deformed Graßmannian is useful for loop calculations!
- Integrability community: work out all Yangian invariants for all reps of $\mathfrak{gl}(N|M)$.

Thank you!