Ambitwistor Strings for Four Dimensions

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Motivation

Since the formulation of the twistor string theories [Witten, Berkovits, Skinner], many remarkable formulae for tree-level scattering amplitudes have been developed [RSVW, ACCK], [Hodges, Cachazo-YG, Cachazo-Skinner, Cachazo-He-Yuan].

This inevitably raises questions regarding the underlying theories: What is the origin of these representations of Yang-Mills and gravity scattering amplitudes?

Recent work has focussed on answering this question, and beyond providing a geometric explanation of the formulae, it also facilitated extensions in various directions. In particular, the CHY [Cachazo-He-Yuan] representation has been understood as arising from string theories in ambitwistor space, the space of null geodesics.

Scattering equations and CHY formulae

[Cachazo-He-Yuan]

- $P(\sigma)$ holomorphic map from Riemann sphere into momentum space,

$$P : \mathbb{CP}^1 \to \mathbb{CP}^d, \qquad P(\sigma) = \sum_j \frac{k_j}{\sigma - \sigma_j}.$$

Scattering equations

$$k_i \cdot P(\sigma_i) = \sum_{j \neq i} \frac{k_i \cdot k_j}{\sigma_i - \sigma_j} = 0$$

- Representation of YM and gravity scattering amplitudes

$$\mathcal{A} = \int \frac{\prod_{i=1}^n d\sigma_i}{\mathrm{Vol}\, \mathrm{SL}(2;\mathbb{C})} \frac{1}{\prod_{i=1}^n \sigma_{i,i+1}} \prod_i{}' \bar{\delta}(k_i \cdot P(\sigma_i)) \, \mathrm{Pf}'(\Psi)$$

$$\mathcal{M} = \int \frac{\prod_{i=1}^n d\sigma_i}{\mathrm{Vol}\, \mathrm{SL}(2;\mathbb{C})} \prod_i{}' \bar{\delta}(k_i \cdot P(\sigma_i)) \, \mathrm{Pf}'(\Psi) \, \mathrm{Pf}'(\tilde{\Psi})$$

The Ambitwistor String in d=10

Ambitwistor space $\mathbb{A}$

Ambitwistor space = space of complex null geodesics in $M_{\mathbb{C}}$

- Symplectic quotient of cotangent bundle of (supersymmetric) spacetime $(X, P, \Psi) \in T^*M$ by constraints $P^2 = 0$ and $\Psi_r \cdot P = 0$

$$\mathbb{A} := \left\{ (X^\mu, P_\mu, \Psi_r^\mu) \in T^*M \,\middle|\, P^2 = 0,\ \Psi_r \cdot P = 0 \right\} \Big/ \{\mathcal{D}_0, \mathcal{D}_r\}$$

  with Hamiltonian vector fields $\mathcal{D}_0 = P \cdot \nabla$, $\mathcal{D}_r = \Psi_r \cdot \nabla + P \cdot \partial_{\Psi_r}$

- $\mathbb{A}$ is a symplectic holomorphic manifold, with symplectic potential

$$\Theta = P \cdot \bar{\partial} X + \frac{1}{2} \sum_r \Psi_r \cdot \mathrm{d}\Psi_r$$

The Ambitwistor String in d=10

RNS ambitwistor string

[Mason-Skinner] (see also [Adamo-Casali-Skinner, Berkovits])

- Complexify action of massless spinning particle

$$S = \frac{1}{2\pi}\int P\cdot\bar\partial X + \frac{1}{2}\sum_r \Psi_r\cdot\bar\partial\Psi_r - \frac{e}{2}P^2 - \chi_r P\cdot\Psi_r$$

  Geometrically, the action is obtained from the symplectic potential Θ, and the gauge fields e and χ_r impose the constraints. This reduces the phase space to $\mathbb{A}$.

- BRST operator $Q = \oint cT + \frac{\tilde c}{2}P^2 + \sum_r \gamma_r P\cdot\Psi_r + \frac{\tilde b}{2}\gamma_r\gamma_r$, nilpotent in $d = 10$ as in the usual superstring

- In particular, the correlation functions of appropriate VO can be shown to yield the CHY formulae.

Four dimensions

In four dimensions, the space of null geodesics has an alternative spinorial representation in addition to the vector representation used in the formulation of the RNS ambitwistor string. This suggests that the ambitwistor string ideas can be implemented naturally to construct models for Yang-Mills and gravity.

These models allow for any amount of supersymmetry, and the correlation functions lead to new, remarkably simple formulae for tree-level scattering amplitudes which are

- supported on the scattering equations
- parity invariant.

Outline

1. Ambitwistor strings in d=4
   - Ambitwistor space
   - Worldsheet Theory
2. Yang-Mills
3. Gravity
4. Ambitwistor Strings at Null Infinity
   - Geometry and Symmetries
   - Worldsheet theory
   - Soft limits

Ambitwistor strings in d=4

Ambitwistor space $\mathbb{A}_4$

Alternative twistorial representation:

$$Z = (\lambda_\alpha, \mu^{\dot\alpha}, \chi^r) \in \mathbb{T} = \mathbb{C}^{4|\mathcal{N}}$$
$$W = (\tilde\mu^\alpha, \tilde\lambda_{\dot\alpha}, \tilde\chi_r) \in \mathbb{T}^* = \mathbb{C}^{4|\mathcal{N}}$$

Ambitwistor space is the quadric $Z \cdot W = 0$ inside $\mathbb{T} \times \mathbb{T}^*$

$$\mathbb{A} := \left\{(Z^I, W_I) \in \mathbb{T} \times \mathbb{T}^* \,|\, Z \cdot W = 0\right\} \Big/ \left\{Z \tfrac{\partial}{\partial Z} - W \tfrac{\partial}{\partial W}\right\},$$

which can be seen as the symplectic quotient of $\mathbb{T} \times \mathbb{T}^*$ by the Hamiltonian $Z \cdot W$. $\mathbb{A}$ is thus a symplectic manifold with the potential

$$\Theta = \frac{i}{2}(W \cdot \mathrm{d}Z - Z \cdot \mathrm{d}W)$$

Ambitwistor strings in d=4

Ambitwistor space $\mathbb{A}_4$

Comments:

- The incidence relations

$$\mu^{\dot\alpha} = i(x^{\alpha\dot\alpha} + i\theta^{a\alpha}\tilde\theta^{\dot\alpha}_a)\lambda_\alpha\,, \qquad \chi^a = \theta^{a\alpha}\lambda_\alpha$$
$$\tilde\mu^{\alpha} = -i(x^{\alpha\dot\alpha} - i\theta^{a\alpha}\tilde\theta^{\dot\alpha}_a)\tilde\lambda_{\dot\alpha}\,, \qquad \tilde\chi_a = \tilde\theta^{\dot\alpha}_a\tilde\lambda_{\dot\alpha}$$

  realize a point in chiral Minkowski space as a quadric, $\mathbb{CP}^1 \times \mathbb{CP}^1$.

- Define $P_{\alpha\dot\alpha} = \lambda_\alpha\tilde\lambda_{\dot\alpha}$, then the null geodesic constraint $P^2 = 0$ (appearing in the vectorial representation) is explicitly solved.

Worldsheet Theory

- Motivation: In analogy to the ambitwistor string in $d = 10$, we will complexify the action of a massless spinning particle, the Ferber superparticle.
- Again, the action S is determined by the symplectic potential Θ, and the constraint $Z \cdot W = 0$ is imposed by introducing a gauge field a;

$$S = \frac{1}{2\pi} \int_\Sigma W \cdot \bar\partial Z - Z \cdot \bar\partial W + a\, Z \cdot W\,.$$

Here, (Z, W) are spinors on the worldsheet,

$$(Z, W) \in \Omega^0(\Sigma, (\mathbb{T} \times K^{1/2}) \times (\mathbb{T}^* \times K^{1/2}))$$

- Adding worldsheet gravity and gauge-fixing yields the BRST operator

$$Q = \oint c(W \cdot \partial Z - Z \cdot \partial W) + uZ \cdot W$$

Note: In general anomalous!

Yang-Mills Amplitudes

Vertex Operators

- Introduce integrated and unintegrated vertex operators for self-dual and anti self-dual fields

$$V'_a = \int \frac{\mathrm{d}s_a}{s_a}\bar{\delta}^2(\lambda_a - s_a\lambda)\mathrm{e}^{is_a\left([\mu\,\tilde{\lambda}_a]+\chi^r\tilde{\eta}_{ar}\right)} j \cdot t_a \qquad \mathcal{V}'_a = \int d\sigma_a V'_a$$

$$\widetilde{V}_a = \int \frac{\mathrm{d}s_a}{s_a}\bar{\delta}^2(\tilde{\lambda}_a - s_a\tilde{\lambda})\mathrm{e}^{is_a(\langle\tilde{\mu}\,\lambda_a\rangle+\tilde{\chi}_r\eta^r_a)} j \cdot t_a \qquad \widetilde{\mathcal{V}}_a = \int d\sigma_a \widetilde{V}_a$$

where j denotes a current algebra, and t_a are Lie algebra elements.

- More convenient representation of the supersymmetry:

$$V_a = \int \frac{\mathrm{d}s_a}{s_a}\bar{\delta}^{2|\mathcal{N}}(\lambda_a - s_a\lambda|\eta_a - s_a\chi)\mathrm{e}^{is_a[\mu\,\tilde{\lambda}_a]} j \cdot t_a\,, \quad \mathcal{V}_a = \int d\sigma_a V_a$$

Yang-Mills Amplitudes

Worldsheet correlation function

N^{k}MHV amplitudes as correlation function

$$\mathcal{A} = \left\langle \widetilde{\mathcal{V}}_1 \ldots \widetilde{\mathcal{V}}_k \mathcal{V}_{k+1} \ldots \mathcal{V}_n \right\rangle .$$

- Take exponential factors appearing in the vertex operators into the action to obtain the effective field equations

$$\bar{\partial}_\sigma Z = \bar{\partial}\left(\lambda, \mu, \chi\right) = \sum_{i=1}^{k} s_i \left(\lambda_i, 0, \eta_i\right) \bar{\delta}\left(\sigma - \sigma_i\right),$$

$$\bar{\partial}_\sigma W = \bar{\partial}\left(\tilde{\mu}, \tilde{\lambda}, \tilde{\chi}\right) = \sum_{p=k+1}^{n} s_p \left(0, \tilde{\lambda}_p, 0\right) \bar{\delta}(\sigma - \sigma_p).$$

- (Z, W) are worldsheet spinors, thus unique solution

$$Z(\sigma) = \sum_{i=1}^{k} \frac{1}{(\sigma\, \sigma_i)} \left(\lambda_i, 0, \eta_i\right) , \quad W(\sigma) = \sum_{p=k+1}^{n} \frac{1}{(\sigma, \sigma_p)} \left(0, \tilde{\lambda}_p, 0\right) .$$

Yang-Mills Amplitudes

$\mathcal{A} = \langle \widetilde{\mathcal{V}}_1 \dots \widetilde{\mathcal{V}}_k \mathcal{V}_{k+1} \dots \mathcal{V}_n \rangle$

Yang-Mills amplitudes

$$\mathcal{A} = \int \frac{\prod_{a=1}^n \mathrm{d}^2\sigma_a}{\mathrm{Vol\,GL}(2,\mathbb{C})} \frac{1}{\prod_{a=1}^n (a\, a{+}1)} \prod_{i=1}^k \bar{\delta}^2(\tilde{\lambda}_i - \tilde{\lambda}(\sigma_i)) \prod_{p=k+1}^n \bar{\delta}^{2|\mathcal{N}}(\lambda_p - \lambda(\sigma_p)) \,.$$

where

$$\lambda(\sigma) = \sum_{i=1}^k \frac{\lambda_i}{(\sigma,\, \sigma_i)}\,, \quad \tilde{\lambda}(\sigma) = \sum_{p=k+1}^n \frac{\tilde{\lambda}_p}{(\sigma, \sigma_p)}\,.$$

Yang-Mills Amplitudes

$\mathcal{A} = \left\langle \widetilde{\mathcal{V}}_1 \dots \widetilde{\mathcal{V}}_k \mathcal{V}_{k+1} \dots \mathcal{V}_n \right\rangle$

Yang-Mills amplitudes

$$\mathcal{A} = \int \frac{\prod_{a=1}^n \mathrm{d}^2\sigma_a}{\mathrm{Vol}\,\mathrm{GL}(2,\mathbb{C})} \frac{1}{\prod_{a=1}^n (a\,a{+}1)} \prod_{i=1}^k \bar{\delta}^2(\tilde{\lambda}_i - \tilde{\lambda}(\sigma_i)) \prod_{p=k+1}^n \bar{\delta}^{2|\mathcal{N}}(\lambda_p - \lambda(\sigma_p))\,.$$

where

$$\lambda(\sigma) = \sum_{i=1}^k \frac{\lambda_i}{(\sigma,\, \sigma_i)}\,, \quad \tilde{\lambda}(\sigma) = \sum_{p=k+1}^n \frac{\tilde{\lambda}_p}{(\sigma, \sigma_p)}\,.$$

In particular, these tree-level scattering amplitudes

- localize fully on the support of the scattering equations
- contain only σ moduli, no additional moduli from the degree d of a line bundle
- are manifestly parity invariant.

The Scattering Equations in $d = 4$

- Momentum conservation

  On support of the scattering equations

$$\sum_{p=k+1}^{n} \lambda_p \tilde{\lambda}_p = \sum_{p=k+1}^{n} \sum_{j=1}^{k} \tilde{\lambda}_p \frac{\lambda_j}{(\sigma_p \, \sigma_j)} = -\sum_{j=1}^{k} \lambda_j \tilde{\lambda}_j,$$

- Scattering Equations

  In twistorial representation:

Twistorial Scattering Equations

$$0 = [\tilde{\lambda}_i, \tilde{\lambda}(\sigma_i)], \qquad i = 1, \ldots, k \qquad \tilde{\lambda}(\sigma) = \sum_{p=k+1}^{n} \frac{\tilde{\lambda}_p}{(\sigma, \sigma_p)}$$

$$0 = \langle \lambda_p, \lambda(\sigma_p) \rangle, \qquad p = k+1, \ldots, n \qquad \lambda(\sigma) = \sum_{i=1}^{k} \frac{\lambda_i}{(\sigma, \sigma_i)}$$

Define $P_{\alpha\dot{\alpha}}(\sigma) = \lambda_\alpha(\sigma)\tilde{\lambda}_{\dot{\alpha}}(\sigma)$, then the twistorial scattering equations imply the (usual) scattering equations

$$\lambda_a^{\alpha} \tilde{\lambda}_a^{\dot{\alpha}} \cdot P_{\alpha\dot{\alpha}}(\sigma_a) = 0$$

Note: Twistorial scattering equations refined by MHV degree.

Proof of the new formula

Comparison to the RSVW formula

These new representations for tree-level scattering amplitudes can be proven by mapping them onto the well-known RSVW formula for $\mathcal{N}=4$ SYM.

- Recall the RSVW formula

$$\mathcal{A}_{\text{RSVW}} = \int \frac{\prod_{r=0}^{d} d^{4|4} Z_r}{\text{Vol GL}(2,\mathbb{C})} \prod_{a=1}^{n} \frac{\mathrm{d}\sigma_a}{(a\, a{+}1)} \prod_{a=1}^{n} A_a(Z)$$

  with momentum eigenstates $A_a(Z)$ and map moduli $Z_r(\sigma) = (\lambda, \mu, \chi)$.
- The equality $\mathcal{A}_{\text{RSVW}} = \mathcal{A}$ is established by integrating out $Z_r(\sigma)$ and a change of variables

$$s_i = \frac{1}{\prod_{l=1, l\neq i}^{k} (i\, l)\, t_i} \qquad i = 1, \dots k$$

$$s_p = \prod_{l=1}^{k} (p\, l)\, t_p \qquad p = k+1, \dots, n.$$

Gravity as an ambitwistor string

Worldsheet Theory

In analogy to the twistor string proposed in [Skinner], we can construct an ambitwistor string theory for Einstein gravity.

- Field content: worldsheet spinors

$$(Z, W) \in \Omega^0(\Sigma, (\mathbb{T} \times K^{1/2}) \times (\mathbb{T}^* \times K^{1/2}))$$
$$(\rho, \tilde{\rho}) \in \Pi\Omega^0(\Sigma, (\mathbb{T} \times K^{1/2}) \times (\mathbb{T}^* \times K^{1/2}))$$

- Breaking conformal invariance:
  Introduce the infinity twistors $\mathcal{I}_{IJ}$, $\mathcal{I}^{IJ}$, which determine a preferred metric on spacetime and encode a cosmological constant.

$$\mathcal{I}_{IJ}\mathcal{I}^{JK} = \Lambda\delta^I_K, \qquad \mathcal{I}_{IJ} = \begin{pmatrix} \epsilon^{\alpha\beta} & 0 \\ 0 & \Lambda\epsilon_{\dot\alpha\dot\beta} \end{pmatrix}, \qquad \mathcal{I}^{IJ} = \begin{pmatrix} \Lambda\epsilon_{\alpha\beta} & 0 \\ 0 & \epsilon^{\dot\alpha\dot\beta} \end{pmatrix}.$$

  In particular, $\text{rank}(\mathcal{I}) = 2$ for $\Lambda = 0$.

Gravity as an ambitwistor string

Worldsheet Theory

- We can now formulate the gravitational action

$$S = \frac{1}{2\pi}\int Z\cdot\bar{\partial}W - W\cdot\bar{\partial}Z + \tilde{\rho}\cdot\bar{\partial}\rho - \rho\cdot\bar{\partial}\tilde{\rho}\,.$$

- For Einstein gravity, we furthermore have the current algebra

$$K_a = (Z\cdot W, \rho\cdot\tilde{\rho}, Z\cdot\tilde{\rho}, W\cdot\rho, \langle Z\rho\rangle, [W\tilde{\rho}], \langle\rho\,\rho\rangle, [\tilde{\rho}\,\tilde{\rho}])\,.$$

Gauging all the currents yields the BRST operator Q, with ghosts (β_a, γ^a) and structure constants C^a_{bc} of the current algebra K_a.

$$Q_{BRST} = \oint cT + \gamma^a K_a - \frac{i}{2}\beta_a\gamma^b\gamma^c C^a_{bc}\,,$$

Gravity Amplitudes

Vertex Operators

As in YM, we obtain integrated and unintegrated vertex operators for sd and asd fields, corresponding to the on-shell pull-back from $\mathbb{T}$ or $\mathbb{T}^*$,

$$V_h = \int_\Sigma \delta^2(\gamma) h\,, \qquad \widetilde{V}_{\tilde h} = \int_\Sigma \delta^2(\nu)\tilde h\,,$$

$$\mathcal{V}_p = \int_\Sigma (1 + \rho\cdot\partial_Z \tilde\rho\cdot\partial_W)\,\frac{\mathrm{d}t_p}{t_p^3}\bar\delta^2(\lambda_p - t_p\lambda(\sigma_p))\,[\tilde\lambda(\sigma_p)\,\tilde\lambda_p]\,\mathrm{e}^{it_p[\mu(\sigma_p)\tilde\lambda_p]}\,,$$

$$\widetilde{\mathcal{V}}_i = \int_\Sigma (1 + \rho\cdot\partial_Z \tilde\rho\cdot\partial_W)\,\frac{\mathrm{d}t_i}{t_i^3}\bar\delta^2(\tilde\lambda_i - t_i\tilde\lambda(\sigma_i))\,\langle\lambda(\sigma_i)\,\lambda_i\rangle\,\mathrm{e}^{it_i\langle\tilde\mu(\sigma_i)\lambda_i\rangle}\,.$$

Gravity Amplitudes

Correlation function

Amplitudes are now given by the worldsheet correlation function

$$\mathcal{M} = \left\langle \widetilde{V}_{\tilde{h}_1} \prod_{i=2}^{k} \widetilde{\mathcal{V}}_{\tilde{h}_i} \prod_{p=k+1}^{n-1} \mathcal{V}_{h_p} V_{h_n} \right\rangle . \qquad (1)$$

- As in YM, solve the equations of motion of the effective action;

$$Z(\sigma) = \sum_{i=1}^{k} \frac{1}{(\sigma\, \sigma_i)} (\lambda_i, 0, \eta_i) \,, \quad W(\sigma) = \sum_{p=k+1}^{n} \frac{1}{(\sigma, \sigma_p)} \left(0, \tilde{\lambda}_p, 0\right) .$$

- For $\Lambda = 0$, no contractions between $\mathcal{V}_p$ and $\widetilde{\mathcal{V}}_i$.
  To perform the calculation, note that the correlator of the fermionic $(\rho, \tilde{\rho})$ system is the determinant of the matrix of possible contractions.

Gravity Amplitudes

Correlation function $\mathcal{M} = \left\langle \widetilde{V}_{\tilde{h}_1} \prod_{i=2}^{k} \widetilde{\mathcal{V}}_{\tilde{h}_i} \prod_{p=k+1}^{n-1} \mathcal{V}_{h_p} V_{h_n} \right\rangle$

Gravity Amplitudes

$$\mathcal{M} = \int \frac{\prod_{a=1}^{n} \mathrm{d}^2\sigma_a}{\mathrm{Vol}\,\mathrm{GL}(2,\mathbb{C})} \det{}'(\mathcal{H}) \prod_{i=1}^{k} \bar{\delta}^2(\tilde{\lambda}_i - \tilde{\lambda}(\sigma_i)) \prod_{p=k+1}^{n} \bar{\delta}^{2|\mathcal{N}}(\lambda_p - \lambda(\sigma_p))$$

where

$$\mathcal{H} = \begin{pmatrix} \mathbb{H} & 0 \\ 0 & \widetilde{\mathbb{H}} \end{pmatrix},$$

and for $i, j \in \{1, ..., k\}$ and $p, q \in \{k+1, ..., n\}$,

$$\mathbb{H}_{ij} = \frac{\langle i\, j\rangle}{(i\, j)}, \quad i \neq j, \qquad \mathbb{H}_{ii} = -\sum_{j=1, j\neq i}^{k} \mathbb{H}_{ij}$$

$$\widetilde{\mathbb{H}}_{pq} = \frac{[p\, q]}{(p\, q)}, \quad p \neq q, \qquad \widetilde{\mathbb{H}}_{pp} = -\sum_{q=k+1, q\neq p}^{n} \widetilde{\mathbb{H}}_{pq}.$$

Proof of the new formula

Comparison to the Cachazo-Skinner formula

- **Outline of the proof**
  As in Yang-Mills, this new representation for tree-level scattering amplitudes can be proven by establishing a correspondence to the Cachazo-Skinner formula for supergravity. The proof follows along the same idea as in YM; integrating out moduli and redefining variables.

- **Link/Grassmannian-like representation**
  Along similar lines, we can prove the equality of this new representation to the Link /Grassmannian-like representations found in [Cachazo-Mason-Skinner]: substitute the momentum eigenstates in the vertex operators by elemental states.

$$f_{Z_i}(Z) = \int \frac{\mathrm{d}s}{s^{2h-1}} \delta^{4|\mathcal{N}}(Z_i - sZ)\,, \qquad f_{W_i}(Z) = \int \frac{\mathrm{d}s}{s^{2h-1}} \exp(sW_i \cdot Z)\,.$$

  These are wave functions that are supported at points or planes in twistor space.

Summary

We have defined new ambitwistor string theories in $d = 4$, leading to simple representations of tree-level scattering amplitudes in YM and gravity with arbitrary degree of supersymmetry,

$$\mathcal{A} = \int \frac{\prod_{i=1}^{n} \mathrm{d}^2\sigma_i}{\mathrm{Vol}\,\mathrm{GL}(2,\mathbb{C})} \frac{1}{\prod_{i=1}^{n}(i\,i{+}1)} \prod_{i=1}^{k} \bar{\delta}^{2|\mathcal{N}}(\lambda_i - \lambda(\sigma_i)) \prod_{a=k+1}^{n} \bar{\delta}^2(\tilde{\lambda}_a - \tilde{\lambda}(\sigma_a))$$

$$\mathcal{M} = \int \frac{\prod_{i=1}^{n} \mathrm{d}^2\sigma_i}{\mathrm{Vol}\,\mathrm{GL}(2,\mathbb{C})} \det{}'(\mathbb{H}) \det{}'(\widetilde{\mathbb{H}}) \prod_{i=1}^{k} \bar{\delta}^{2|\mathcal{N}}(\lambda_i - \lambda(\sigma_i)) \prod_{a=k+1}^{n} \bar{\delta}^2(\tilde{\lambda}_a - \tilde{\lambda}(\sigma_a)).$$

These formulae for $\mathcal{A}$ and $\mathcal{M}$

- localize on support of scattering equations
- depend on very few moduli
- are ambidextrous, and manifestly parity invariant.

Ambitwistor Strings at Null Infinity

Motivation

Recall that ambitwistor space is defined as the phase space of complex null geodesics, and can thus be formulated over any Cauchy hypersurface; in particular, $\mathbb{A}$ is identified with the cotangent bundle of the hypersurface.

The S-matrix is, almost by definition, a holographic object, defined in terms of asymptotic states. It is thus suggestive to try to formulate the ambitwistor string for an asymptotically flat spacetime with respect to null infinity $\mathscr{I}$. In particular, this implies that ambitwistor space is identified with the cotangent bundle of (complexified) null infinity $\mathbb{A} = T^*\mathscr{I}$.

BMS symmetries

This is of particular interest considering the recent work in this area: [Strominger et.al.] identified the soft limits of scattering amplitudes as Ward identities associated to the BMS symmetries of asymptotically flat spacetimes, and [Adamo-Casali-Skinner] proposed a 2d CFT on $\mathscr{I}$ realizing this correspondence.

The BMS group is the group of asymptotic symmetries at null infinity $\mathscr{I} \cong \mathbb{R} \times S^2$, and consists of

- supertranslations
- (super)rotations

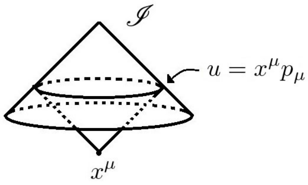

Figure : Diagram of null infinity, $\mathscr{I}$.

From BMS to soft limits

[Strominger et.al.]

- Soft theorems as Ward identities associated with the diagonal subgroup of $BMS^+ \otimes BMS^-$:

$$\langle out| \, B^+ \mathcal{S} - \mathcal{S} B^- \, |in\rangle = 0,$$

where $B^\pm$ are extended BMS generators acting at $\mathscr{I}^\pm$.

- The soft gravitons emerge as Goldstone bosons; specializing on supertranslations $T^\pm$,

$$T^- \, |in\rangle = F^- \, |in\rangle + \textstyle\sum_{k \in in} E_k f\left(z_k, \bar{z}_k\right) |in\rangle$$
$$\langle out| \, T^+ = \langle out| \, F^+ + \textstyle\sum_{j \in out} E_j f\left(z_j, \bar{z}_j\right) \langle out|$$

where $F^\pm$ are outgoing/incoming soft graviton operators. This yields directly the Weinberg soft graviton theorem:

$$\langle out| \, F^+ \mathcal{S} - \mathcal{S} F^- \, |in\rangle = \Big(\textstyle\sum_k E_k f\left(z_k, \bar{z}_k\right) - \sum_j E_j f\left(z_j, \bar{z}_j\right) \Big) \langle out| \, \mathcal{S} \, |in\rangle \, .$$

Review: Soft Limits

[Weinberg, Cachazo-Strominger, Casali, ...]

In the soft limit, gravity scattering amplitudes behave as

$$\mathcal{M}_{n+1} = \left(S^{(0)} + S^{(1)} + S^{(2)}\right)\mathcal{M}_n\,,$$

where

$$S^{(0)} = \sum_{a=1}^{n} \frac{[as]\langle \xi\, a\rangle^2}{\langle a\, s\rangle\langle \xi\, s\rangle^2},$$

$$S^{(1)} = \sum_{a=1}^{n} \frac{[a\, s]\langle \xi\, a\rangle}{\langle a\, s\rangle\langle \xi\, s\rangle}\tilde{\lambda}_s \cdot \frac{\partial}{\partial\tilde{\lambda}_a},$$

$$S^{(2)} = \frac{1}{2}\sum_{a=1}^{n} \frac{[a\, s]}{\langle a\, s\rangle}\tilde{\lambda}_s^{\dot\alpha}\tilde{\lambda}_s^{\dot\beta}\frac{\partial^2}{\partial\tilde{\lambda}_a^{\dot\alpha}\partial\tilde{\lambda}_a^{\dot\beta}}\,.$$

Review: Soft Limits

[Weinberg, Cachazo-Strominger, Casali, ...]

Similarly, for Yang-Mills, the soft limits are given by

$$\mathcal{A}_{n+1} = \left(S^{(0)} + S^{(1)}\right)\mathcal{A}_n \,, \tag{2}$$

where

$$S^{(0)}_{ym} = \frac{\langle 1\, n\rangle}{\langle s\, 1\rangle\langle s\, n\rangle}$$

$$S^{(1)}_{ym} = \frac{1}{\langle s1\rangle}\tilde{\lambda}_s \cdot \frac{\partial}{\partial\tilde{\lambda}_1} + \frac{1}{\langle ns\rangle}\tilde{\lambda}_s \cdot \frac{\partial}{\partial\tilde{\lambda}_n}$$

Four dimensional ambitwistor space at $\mathscr{I}$

- In four dimensions, no additional coordinates are required in the twistorial representation to implement $\mathbb{A} = T^*\mathscr{I}$. Thus, $\mathbb{A}$ is still represented as the quadric

$$\mathbb{A} = \{(Z, W) \in \mathbb{T} \times \mathbb{T}^* | Z \cdot W = 0\}/\{Z \cdot \partial_Z - W \cdot \partial_W\},$$

and the symplectic potential is given by

$$\Theta = \frac{i}{2}(Z \cdot \mathrm{d}W - W \cdot \mathrm{d}Z).$$

- Introducing again coordinates $(u, p_{\alpha\dot\alpha})$ on $\mathscr{I}$, the projection of $\mathbb{A}$ to $\mathscr{I}$ is implemented by

$$u = -i\langle\lambda\tilde\mu\rangle, \qquad \tilde u = i[\tilde\lambda, \mu], \qquad p_{\alpha\dot\alpha} = \lambda_\alpha\tilde\lambda_{\dot\alpha},$$

In particular note that for $\mathcal{N} = 0$, the ambitwistor constraint $Z \cdot W = 0$ implies $u = \tilde u$.

Geometry and Symmetries

All diffeomorphisms of a manifold have a hamiltonian lift to the cotangent bundle, so in particular all symmetries of $\mathscr{I}$ lift to $\mathbb{A} = T^*\mathscr{I}$.

- Supertranslations: $H_f = f(\lambda, \tilde{\lambda})$, for f of weight $(1,1)$.
- Superrotations $H_r = [\mu, \tilde{r}] + \langle \tilde{\mu}, r \rangle$, where r_α and $\tilde{r}_{\dot{\alpha}}$ are of weight $(1,0)$ and $(0,1)$ respectively

Consider the supertranslations: H_f generates the transformations

$$\delta\tilde{\mu}^\alpha = i\frac{\partial f}{\partial \lambda_\alpha}\,, \quad \text{so} \quad \delta u = \lambda_\alpha \frac{\partial f}{\partial \lambda_\alpha} = f\,,$$

as claimed above.

Geometry and Symmetries

As in the discussion above, we will introduce further fields $(\rho, \tilde{\rho})$ to describe Einstein gravity, and gauge the currents

$$\rho \cdot \tilde{\rho} = Z \cdot \tilde{\rho} = W \cdot \rho = \langle Z\rho \rangle = [W\tilde{\rho}] = \langle \rho\,\rho \rangle = [\tilde{\rho}\,\tilde{\rho}] = 0$$

As before, the symplectic potential the becomes

$$\Theta = \frac{i}{2}\left(Z \cdot \mathrm{d}W - W \cdot \mathrm{d}Z + \rho \cdot \mathrm{d}\tilde{\rho} - \tilde{\rho}\,\mathrm{d}\rho\right) .$$

Extend the Hamiltonians to commute with these constraints by including a factor of $1 + \rho \cdot \partial_Z \tilde{\rho} \cdot \partial_W$, thus giving

- supertranslations $(1 + \rho \cdot \partial_Z \tilde{\rho} \cdot \partial_W)\, H_f$
- superrotations $(1 + \rho \cdot \partial_Z \tilde{\rho} \cdot \partial_W)\, H_r$

Worldsheet theory

As before, the theory is constructed from the symplectic potential, so we get the following action for Yang-Mills:

$$S = \frac{1}{2\pi} \int Z \cdot \bar{\partial} W - W \cdot \bar{\partial} Z + aZ \cdot W .$$

Introducing the additional $(\rho, \tilde{\rho})$ system, the action for gravity is

$$S = \frac{1}{2\pi} \int Z \cdot \bar{\partial} W - W \cdot \bar{\partial} Z + \tilde{\rho} \cdot \bar{\partial} \rho - \rho \cdot \bar{\partial} \tilde{\rho} + e^a K_a \, .$$

The amplitude calculations reduce trivially to those of the original four-dimensional ambitwistor string, and thus yield the expected tree-level scattering amplitudes.

Symmetries and Diffeomorphisms

Note that basing the action on the symplectic potential implies in particular the singular parts of OPE of operators in the ambitwistor string theory precisely arise from the Poisson structure. Hamiltonians h generating diffeomorphisms on $\mathbb{A}$ preserve the symplectic potential, and thus define operators via

$$Q_h = \frac{1}{2\pi i} \oint h.$$

Operators defined in this way generate symplectic diffeomorphisms in the ambitwistor string model.

In particular, all BMS transformations have a Hamiltonian lift to $\mathbb{A} = T^*\mathscr{I}$, and thus define charges of the ambitwistor string model.

Soft limits

The general idea is now to expand the vertex operators in the soft limit. The leading and subleading terms in the expansion can then be identified as generators of supertranslations and superrotations. All further contributions generate diffeomorphisms of $\mathbb{A}$, which will not correspond to diffeomorphisms of $\mathscr{I}$.

Gravity

Using $\bar\delta(\langle\lambda_s\,\lambda(\sigma_s)\rangle) = \bar\partial\frac{1}{2\pi i\langle\lambda_s\,\lambda(\sigma_s)\rangle}$, the soft vertex operator can be written as

$$\begin{aligned}\mathcal{V}_s &= \oint (1+\rho\cdot\partial_Z\tilde\rho\cdot\partial_W)\,\frac{\langle\xi\,\lambda(\sigma_s)\rangle^2[\tilde\lambda(\sigma_s)\,\tilde\lambda_s]}{\langle\xi\,\lambda_s\rangle^2\langle\lambda_s\,\lambda(\sigma_s)\rangle}\,\mathrm{e}^{\left(i\frac{\langle\xi\,\lambda_s\rangle[\mu(\sigma_s)\tilde\lambda_s]}{\langle\xi\,\lambda(\sigma_s)\rangle}\right)}\\ &= \mathcal{V}_s^0+\mathcal{V}_s^1+\mathcal{V}_s^2+\dots\end{aligned}$$

In this expansion,

$$\begin{aligned}
\mathcal{V}_s^0 &= \oint (1+\rho\cdot\partial_Z\tilde{\rho}\cdot\partial_W)\,\frac{\langle\xi\,\lambda(\sigma_s)\rangle^2[\tilde{\lambda}(\sigma_s)\,\tilde{\lambda}_s]}{\langle\xi\,\lambda_s\rangle^2\langle\lambda_s\,\lambda(\sigma_s)\rangle} \\
\mathcal{V}_s^1 &= \oint (1+\rho\cdot\partial_Z\tilde{\rho}\cdot\partial_W)\,\frac{i\langle\xi\,\lambda(\sigma_s)\rangle[\tilde{\lambda}(\sigma_s)\,\tilde{\lambda}_s][\mu(\sigma_s)\tilde{\lambda}_s]}{\langle\xi\,\lambda_s\rangle\langle\lambda_s\,\lambda(\sigma_s)\rangle} \\
\mathcal{V}_s^2 &= \oint (1+\rho\cdot\partial_Z\tilde{\rho}\cdot\partial_W)\,\frac{[\tilde{\lambda}(\sigma_s)\,\tilde{\lambda}_s][\mu(\sigma_s)\tilde{\lambda}_s]^2}{\langle\lambda_s\,\lambda(\sigma_s)\rangle}
\end{aligned}$$

Indeed, we can identify $\mathcal{V}_s^0$ as a supertranslation generator, and $\mathcal{V}_s^1$ as a superrotation generator.
Inserting these contributions into the correlation function yields the soft theorems for gravity scattering amplitudes ($i=0,1,2$),

$$\left\langle\widetilde{\mathcal{V}}_1\ldots\widetilde{\mathcal{V}}_k\mathcal{V}_{k+1}\ldots\mathcal{V}_n\mathcal{V}_s^i\right\rangle = S^{(i)}\left\langle\widetilde{\mathcal{V}}_1\ldots\widetilde{\mathcal{V}}_k\mathcal{V}_{k+1}\ldots\mathcal{V}_n\right\rangle .$$

Summary

Identifying $\mathbb{A} = T^*\mathscr{I}$ with the cotangent bundle at null infinity, symplectic diffeomorphisms of $\mathscr{I}$ define charges of the ambitwistor string (since the action is based on the symplectic potential). When expanding vertex operators in the ambitwistor string in the soft limit, the leading and subleading term can then be identified as generators of supertranslations and superrotations.

Ambitwistor strings at null infinity therefore confirm the relation between Ward identities of BMS symmetries and soft limits.

$$\left\langle \widetilde{\mathcal{V}}_1 ... \widetilde{\mathcal{V}}_k \mathcal{V}_{k+1} ... \mathcal{V}_n \mathcal{V}_s^i \right\rangle = S^{(i)} \left\langle \widetilde{\mathcal{V}}_1 ... \widetilde{\mathcal{V}}_k \mathcal{V}_{k+1} ... \mathcal{V}_n \right\rangle .$$

Outlook and further directions

Ambitwistor string

- Non-zero cosmological constant
  The formulation of the gravitational ambitwistor string model in four dimensions suggests a natural extension to non-zero cosmological constant $\Lambda \neq 0$. Recall in this context that we obtained the theory for a flat spacetime as the degenerate limit of the infinity twistor $\mathcal{I}$,

$$\mathcal{I}_{IJ}\mathcal{I}^{JK} = \Lambda\delta^I_K, \qquad \mathcal{I}_{IJ} = \begin{pmatrix} \epsilon^{\alpha\beta} & 0 \\ 0 & \Lambda\epsilon_{\dot\alpha\dot\beta} \end{pmatrix}, \qquad \mathcal{I}^{IJ} = \begin{pmatrix} \Lambda\epsilon_{\alpha\beta} & 0 \\ 0 & \epsilon^{\dot\alpha\dot\beta} \end{pmatrix}.$$

- Comparison to the RNS ambitwistor string and the CHY formulae
  The correspondence to the CHY formulae is best established at the level of the correlators. On the support of the delta-functions,
  - the ambitwistor Hodges matrix $\det'(\mathcal{H})$ corresponds to one copy of the Pfaffian $\mathrm{Pf}'(\Psi)$.
  - $P(\sigma) := \sum \frac{k_i}{\sigma-\sigma_i} = \lambda(\sigma)\tilde\lambda(\sigma)$.
  - the $\rho\tilde\rho$ system can be understood as a spin representation of the $\Psi\Psi$ current algebra.

Outlook and further directions

Ambitwistor string and Null infinity

- Loop amplitudes: In the models presented here, anomalies pose an obstruction to the extension to loop amplitudes. However, a critical anomaly-free model exists in $d = 10$: the RNS ambitwistor string. It is thus likely that an anomlay free theory can be formulated by coupling to appropriate matter, and dimensional reduction should provide a guideline for its derivation.
- Conformal Gravity: It is possible to supplement the Yang-Mills ambitwistor string by (non-minimal) conformal gravity vertex operators. It might be interesting to try investigate the possibility of a formulation for the minimal model by introducing the $(\rho, \tilde{\rho})$ system familiar from the Einstein gravity case, and only gauging the currents not involving the infinity twistor.
- Ambitwistor string at null infinity: Vertex operator algebra for the ambitwistor string at null infinity, both for general d and in $d = 4$

Thank you!