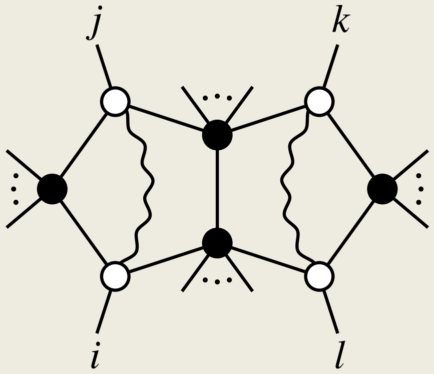
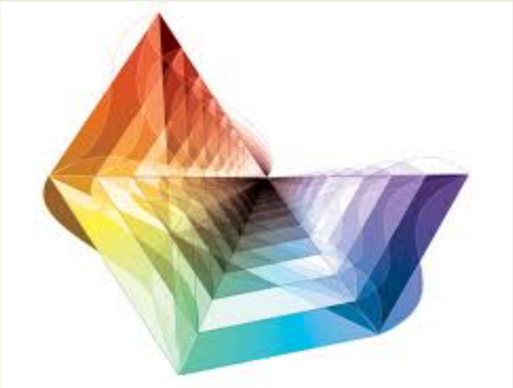
Landau Singularities from the Amplituhedron



Anastasia Volovich
Brown University

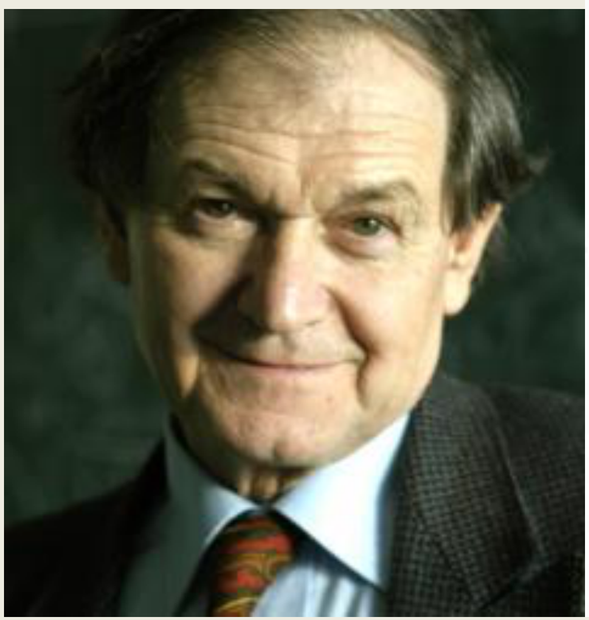


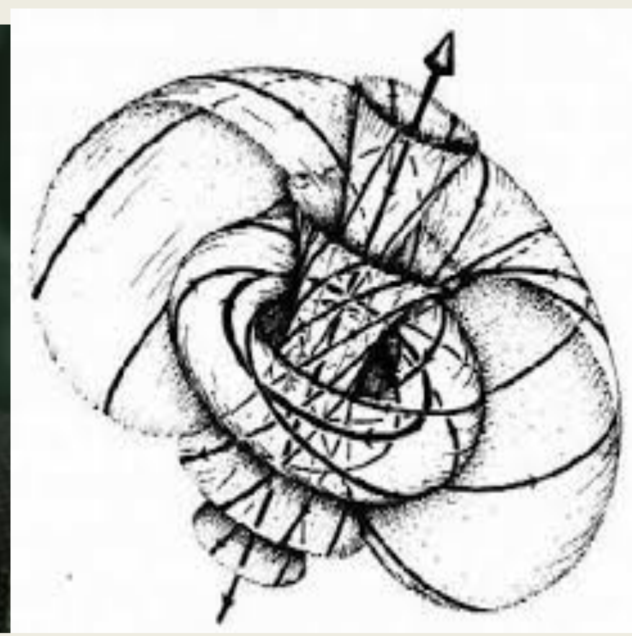
Oxford, January 2017

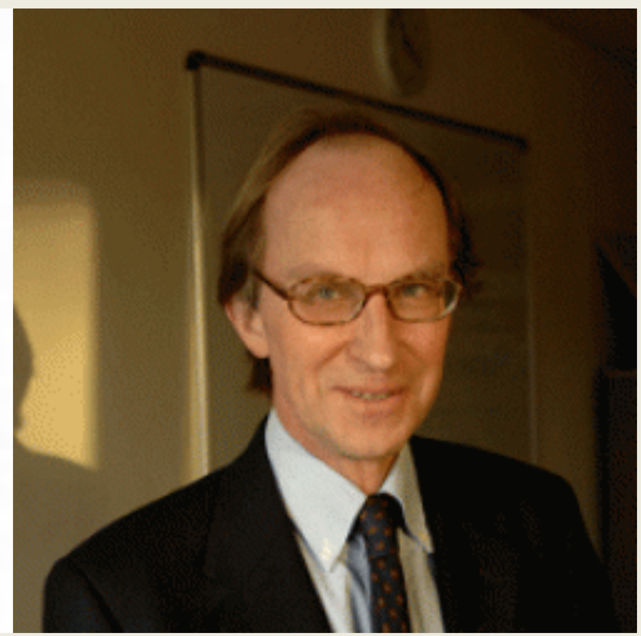
1612.02708

with Tristan Dennen, Igor Prlina, Marcus Spradlin, Stefan Stanojevic

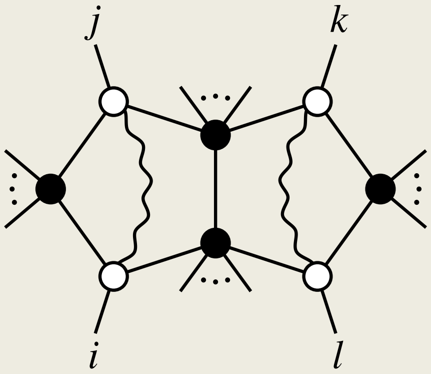

HAPPY
BIRTHDAY







Landau Singularities from the Amplituhedron



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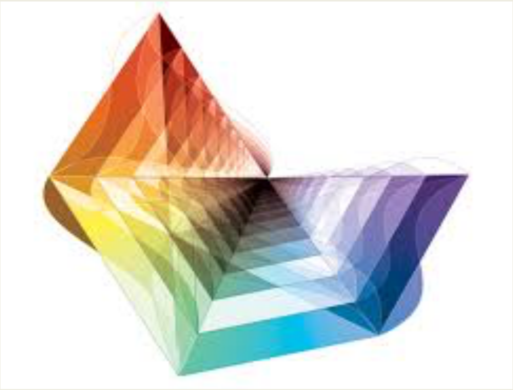


Oxford, January 2017

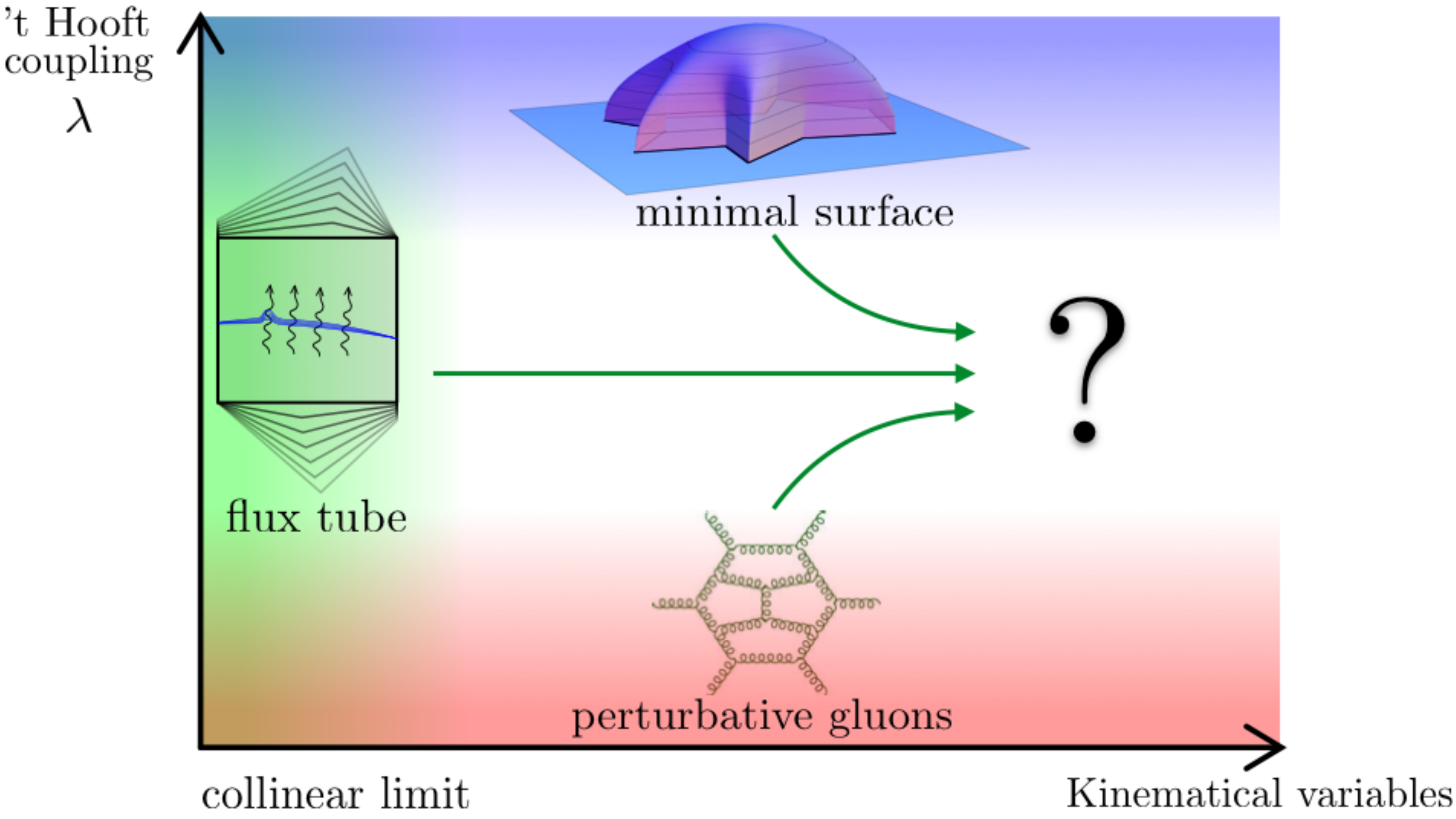
1612.02708

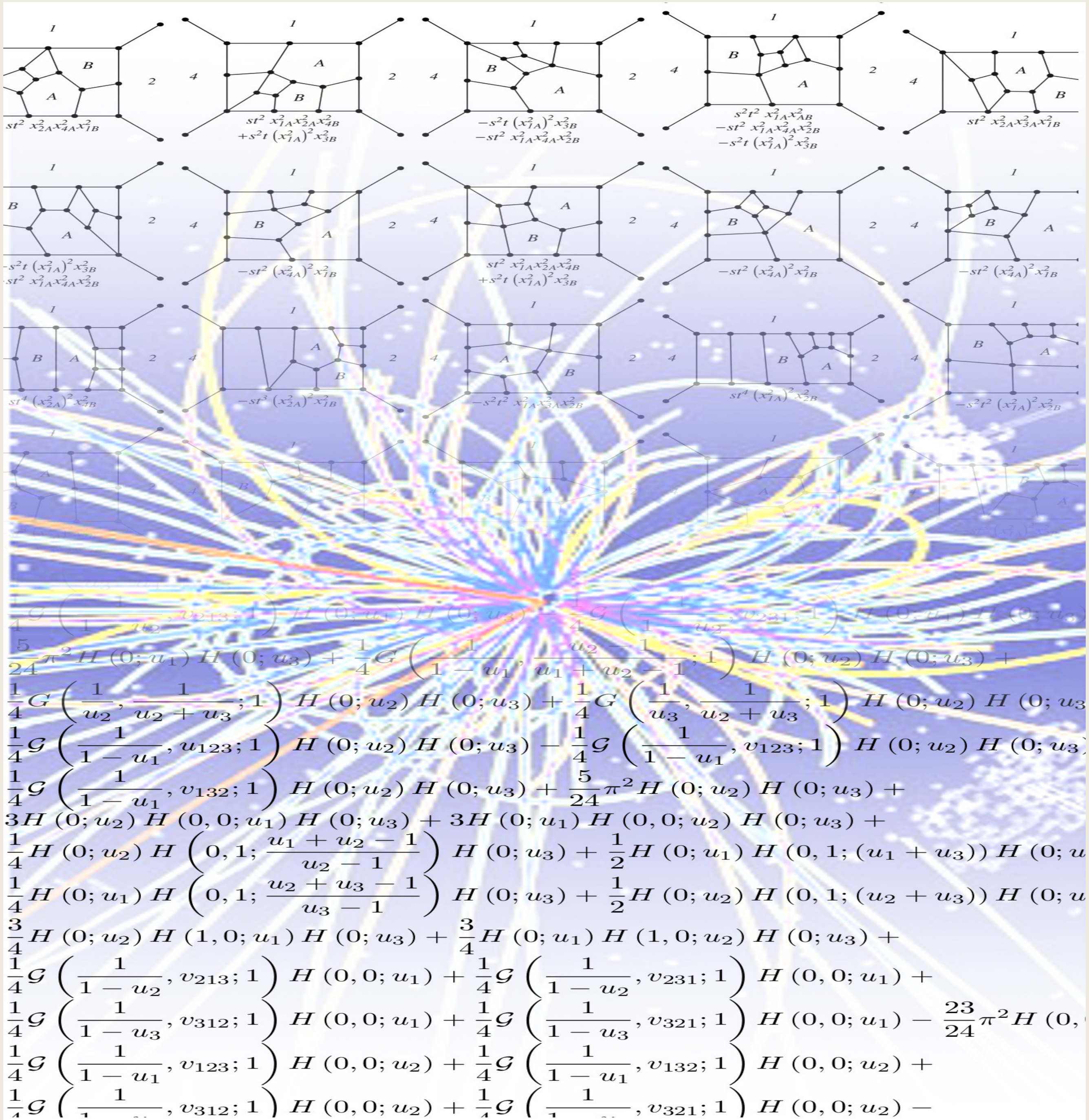
with Tristan Dennen, Igor Prlina, Marcus Spradlin, Stefan Stanojevic

Introduction

- We believe that N=4 Yang-Mills is “integrable” or “solvable”. Some pieces of the theory, such as planar anomalous dimension, can reasonably be said to have been already “solved”.
- But for more complicated quantities, such as general correlation functions or scattering amplitudes, it is not yet completely clear what form a “solutions” will even take.
- This question is being explored “experimentally” from a variety of complementary approaches.

N=4 Yang-Mills





Integrand

Scattering Amplitude

Data is Hard to Come by

- Despite recent advances, relatively few scattering amplitudes in N=4 Yang-Mills are available in the literature.
- 6-point MHV and NMHV up to 5-loops [Caron-Huot, Dixon, McLeon, Von Hippel 2016]
- All 2-loop MHV [Caron-Huot 2011]
- 7-point 2-loop NMHV [Caron-Huot, He 2011]
- 7-point 3-loop MHV [Drummond, Papathanasiou, Spradlin 2014]
- 7-point 4-loop MHV and 7-point 3-loop NMHV [Dixon, Drummond, Harrington, McLeod, Papathanasiou, Spradlin, 2016]

Data is Hard to Come by

- Despite recent advances, relatively few scattering amplitudes in N=4 Yang-Mills are available in the literature.
- We are looking for new tools — like those which exist at tree level — to make loop calculations trivial...
- Moreover, having more “data” in hand is crucial for identifying new, hidden mathematical properties of these amplitudes, and ... ultimately, to get some clues about “what is the class of functions whose perturbative tails we are seeing?”
- It would be enormously valuable to close the gap between our understanding integrands and amplitudes.

S-Matrix Program: Old and New

It has long been a goal of the S-matrix program to be able to construct scattering amplitudes based on a few physical principles and a thorough understanding of their analytic structure.

In todays talk: I will review some technology (Landau singularities) from ELOP (1966).

S-matrix program: Old and New

- Why couldn’t this talk been given 50 years ago?
- We now appreciate that the best arena for carrying out S-matrix program is N=4 Yang-Mills.

SUPERSYMMETRIC YANG-MILLS THEORIES *

Lars BRINK ** and John H. SCHWARZ
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J. SCHERK
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Received 22 December 1976

- OK, why couldn’t this talk been given 40 years ago?

S-Matrix program: New Ingredients

Three relatively recent developments are crucial ingredients:

1. Kinematic Variables: momentum twistors

[Penrose, Hodges 2009]

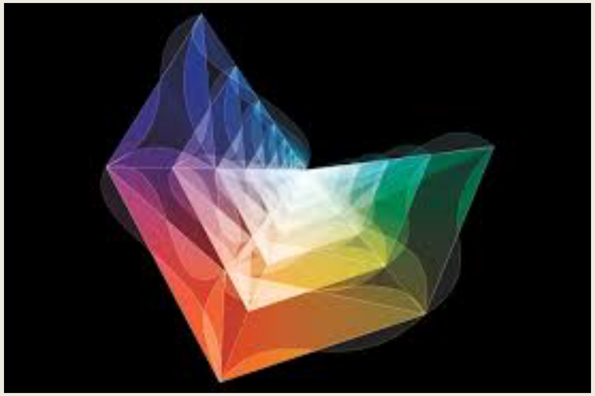
2. Integrals: Symbol of an amplitude

[Goncharov, Spradlin, Vergu, AV 2010]

3. Integrands: Amplituhedron

[Arkani-Hamed, Hodges, Trnka 2013]

Today's Talk



- I will describe a geometric algorithm to determine physical singularities of amplitudes in N=4 Yang-Mills from the amplituhedron.
- I will apply the algorithm to the one- and two-loop MHV amplitudes.
- This is a step towards translating integrands directly to amplitudes.

Dennen, Prlina, Spradlin, Stanojevic, AV

Plan

- Review: Landau equations (ELOP)
- Landau singularities for one and two-loop integrals in N=4 Yang-Mills
- Landau singularities and amplitudes: physical and spurious singularities
- Refined analysis using Amplituhedron
- Singularities from Amplituhedron: geometric algorithm
- Conclusions and open questions

Landau Singularities

Landau equations for a given Feynman integral are a set of kinematic constraints that are necessary for the appearance of a pole or branch point in the integrated function

$$I = c\int \prod_{r=1}^{L} d^D\ell_r \int_{\alpha_i\geq 0} d^\nu\alpha\,\delta(1-\textstyle\sum_{i=1}^{\nu}\alpha_i)\dfrac{\mathcal{N}(\ell_r^\mu, p_i^\mu,\ldots)}{\mathcal{D}^\nu}$$

$$\mathcal{D} = \sum_{i=1}^{\nu}\alpha_i(q_i^2-m_i^2)\,,$$

In this talk: only focus on singularities described by Landau equation

Landau Equations

$$\sum_{i\in\text{loop}}\alpha_i q_i^\mu = 0 \quad \forall\ \text{loops},$$

$$\alpha_i(q_i^2-m_i^2) = 0 \quad \forall i.$$

Landau 1959
Eden, Landshoff, Olive, Polkinghorne
"The Analytic S-Matrix"

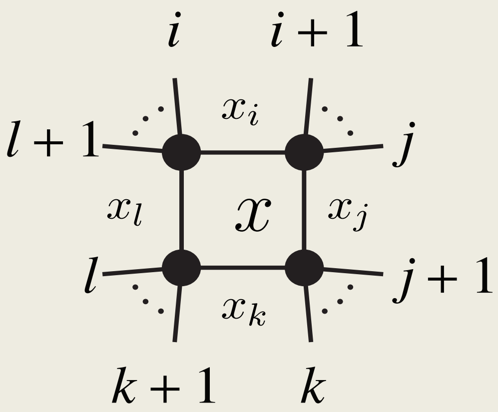
Landau Singularities
locus in external kinematic data
where Landau equations admit solutions

Leading LS	all $\alpha_i \neq 0$	LLS
Subleading LS	some $\alpha_i = 0$	$SLLS$, S^2LLS etc

One-Loop Box

Landau 1959
Eden, Landshoff, Olive, Polkinghorne
"The Analytic S-Matrix"

The Landau equations are easily solved for one-loop box integrals in four dimensions.
(and bubbles and triangles)



$$p_i^\mu = x_i^\mu - x_{i-1}^\mu$$

$$x_{ij}^2 \equiv (x_i - x_j)^2$$

$$(x_i - x_j)^2 = (p_{i+1} + p_{i+2} + \cdots + p_j)^2$$

The second Landau equation puts propagators on-shell (no constraints on external kinematics).

$$(x - x_i)^2 = 0, \quad (x - x_j)^2 = 0, \quad (x - x_k)^2 = 0, \quad (x - x_l)^2 = 0$$

The solvability of the first equation gives a determinant constraint.

$$\alpha_i(x - x_i) + \alpha_j(x - x_j) + \alpha_k(x - x_k) + \alpha_l(x - x_l) = 0$$

Leading Landau Singularities

$$\boxed{0 = (x_{ij}^2 x_{kl}^2 - x_{ik}^2 x_{jl}^2 + x_{il}^2 x_{jk}^2)^2 - 4 x_{ij}^2 x_{jk}^2 x_{kl}^2 x_{il}^2}$$

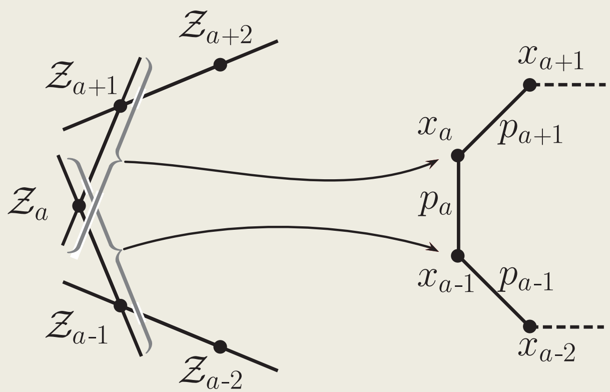
For generic integrals it becomes a hard problem, so next we focus on specific N=4 SYM integrals.

Momentum Twistors

Penrose, Hodges
Arkani-Hamed, Bourjaily, Cachazo, Trnka

Null momentum

$$p_a^\mu \mapsto (p_a)_{\underline{\alpha}\,\underline{\dot\alpha}} \equiv p_a^\mu (\sigma_\mu)_{\underline{\alpha\dot\alpha}} \equiv \lambda^{(a)}_{\underline{\alpha}} \widetilde{\lambda}^{(a)}_{\underline{\dot\alpha}}$$



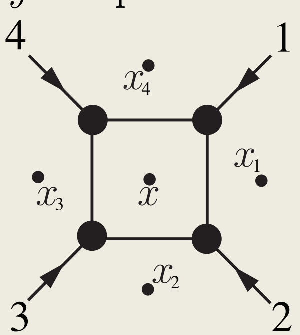
Momentum conservation

$$p_a \equiv x_a - x_{a-1}$$

$$Z = (\lambda, \mu) = (\lambda_\alpha, x_{\alpha\dot\alpha}\lambda^\alpha)$$

$$\langle A\,B\,C\,D\rangle \equiv \epsilon_{IJKL} Z_A^I Z_B^J Z_C^K Z_D^L$$

If x, y are points in Minkowski space associated to two lines $(A,B), (C,D)$ in $\mathbb{P}^3$



$$= \int d^4x \frac{N}{(x-x_1)^2(x-x_2)^2(x-x_3)^2(x-x_4)^2}$$

$$\int_{AB} \frac{\langle 1234\rangle^2}{\langle AB\,12\rangle\langle AB\,23\rangle\langle AB\,34\rangle\langle AB\,41\rangle}$$

$$(x-y)^2 = \frac{\langle A\,B\,C\,D\rangle}{\langle I\,A\,B\rangle\langle I\,C\,D\rangle}$$

Momentum twistors simplify the problem of analyzing solutions to Landau equations.

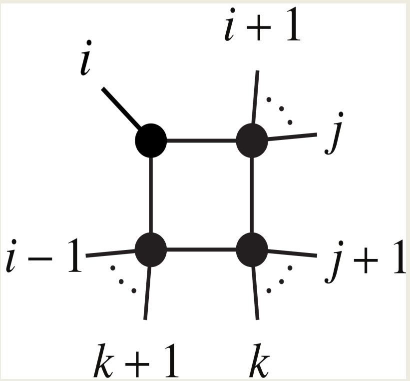
Momentum Twistors! (example)

$$\langle 1\,4\,5\,6\rangle \sim \frac{(p_1+p_2)^2(p_3+p_4)^2}{(p_2+p_3+p_4)^2}\,\frac{u_1+u_2+u_3-1+\sqrt{\Delta}}{2u_1u_2u_3},$$

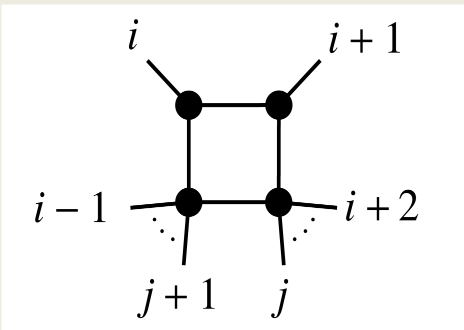
$$\Delta = (1-u_1-u_2-u_3)^2 - 4u_1u_2u_3$$

$$u_1 = \frac{(p_1+p_2)^2(p_4+p_5)^2}{(p_1+p_2+p_3)^2(p_4+p_5+p_6)^2}, \qquad u_2, u_3 = \text{cyclic}$$

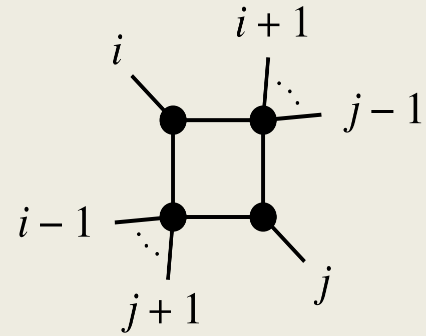
One-loop boxes



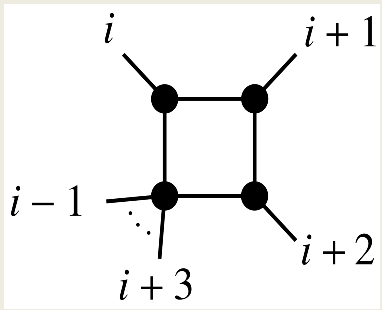
(b)



(c)



(d)



(e)

LLS

Box (b): $0 = \langle i(i{-}1, i{+}1)(j, j{+}1)(k, k{+}1)\rangle$

Box (c): $0 = \langle i{-}1\, i\, i{+}1\, i{+}2\rangle\langle i\, i{+}1\, j\, j{+}1\rangle$

Box (d): $0 = \langle i\, \bar{j}\rangle\langle \bar{i}\, j\rangle$

Box (e): $0 = \langle i{-}1\, i\, i{+}1\, i{+}2\rangle\langle i\, i{+}1\, i{+}2\, i{+}3\rangle$

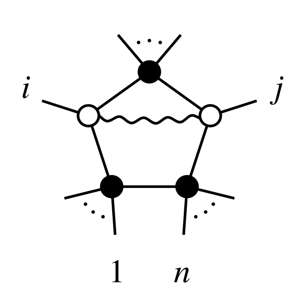
$\bar{a}$ is the plane $(a-1, a, a+1)$

$$\langle C(A,B)(D,E)(G,H)\rangle \equiv \langle (A,B,C)\cap(D,E,C)\, G\, H\rangle$$

$$\langle (A,B,C)\cap(D,E,F)\, G\, H\rangle = \langle A\, B\, C\, G\rangle\langle D\, E\, F\, H\rangle - \langle A\, B\, C\, H\rangle\langle D\, E\, F\, G\rangle$$

One-loop n-point MHV in N=4 SYM

$$\frac{\mathcal{A}^{1-\text{loop}}_{\text{MHV}}}{\mathcal{A}^{\text{tree}}_{\text{MHV}}} = \int_{AB} \sum_{1<i<j<n}$$



Bern, Dixon, Dunbar, Kosower

Arkani-Hamed, Bourjaily, Cachazo, Trnka

chiral pentagon

Dennen, Spradlin, AV

$$\boxed{\frac{\langle AB\,\bar{i}\cap\bar{j}\rangle\langle i\,j\,n\,1\rangle}{\langle AB\,i{-}1\,i\rangle\langle AB\,i\,i{+}1\rangle\langle AB\,j{-}1\,j\rangle\langle AB\,j\,j{+}1\rangle\langle AB\,n\,1\rangle}}$$

(LLS)

$$\langle i\,j\,n\,1\rangle\langle n\,1\,\bar{i}\cap\bar{j}\rangle = 0$$

$(A,B)=(i,j)$ or $(A,B)=\bar{i}\cap\bar{j}$

$$\langle AB\,n\,1\rangle = 0$$

(SLLS)

$$\begin{aligned}
&\langle j(j{-}1,j{+}1)(i,i{+}1)(n,1)\rangle = 0\,,\\
&\langle j(j{-}1,j{+}1)(i{-}1,i)(n,1)\rangle = 0\,,\\
&\langle i(i{-}1,i{+}1)(j,j{+}1)(n,1)\rangle = 0\,,\\
&\langle i(i{-}1,i{+}1)(j{-}1,j)(n,1)\rangle = 0\,,\\
&\langle \bar{i}j\rangle\langle i\bar{j}\rangle = 0\,.
\end{aligned}$$

Reduces to boxes

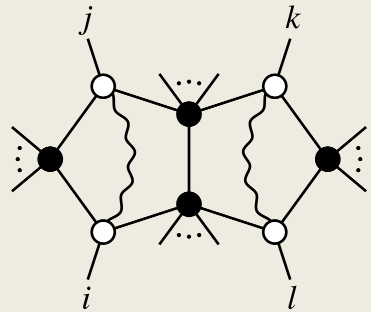
($\mathbf{S^2}$LLS)

$$\begin{aligned}
&\langle i{-}1\;i\;j{-}1\;j\rangle\langle j{-}1\;j\;n\;1\rangle\langle n\;1\;i{-}1\;i\rangle = 0\,,\\
&\langle i\;i{+}1\;j{-}1\;j\rangle\langle j{-}1\;j\;n\;1\rangle\langle n\;1\;i\;i{+}1\rangle = 0\,,\\
&\langle i{-}1\;i\;j\;j{+}1\rangle\langle j\;j{+}1\;n\;1\rangle\langle n\;1\;i{-}1\;i\rangle = 0\,,\\
&\langle i\;i{+}1\;j\;j{+}1\rangle\langle j\;j{+}1\;n\;1\rangle\langle n\;1\;i\;i{+}1\rangle = 0\,.
\end{aligned}$$

Reduces to triangles

Two-loop n-point MHV in N=4 SYM

$$\frac{\mathcal{A}^{\text{2-loop}}_{\text{MHV}}}{\mathcal{A}^{\text{tree}}_{\text{MHV}}} = \int_{AB}\int_{CD} \frac{1}{2}\sum_{i<j<k<l<i}$$



Arkani-Hamed, Bourjaily, Cachazo, Trnka

$$\frac{\langle i\,j\,k\,l\rangle}{\langle ABCD\rangle}\frac{\langle AB\,\bar{i}\cap\bar{j}\rangle}{\langle AB\,i{-}1\,i\rangle\langle AB\,i\,i{+}1\rangle\langle AB\,j{-}1\,j\rangle\langle AB\,j\,j{+}1\rangle}\frac{\langle CD\,\bar{k}\cap\bar{l}\rangle}{\langle CD\,k{-}1\,k\rangle\langle CD\,k\,k{+}1\rangle\langle CD\,l{-}1\,l\rangle\langle CD\,l\,l{+}1\rangle}$$

(LLS)

$$\langle i\;j\;k\;l\rangle\langle i\;j\;\bar{k}\cap\bar{l}\rangle\langle\bar{i}\cap\bar{j}\;k\;l\rangle\langle\bar{i}\cap\bar{j}\;\bar{k}\cap\bar{l}\rangle = 0 \qquad (A,B)=(i,j)\text{ or }\bar{i}\cap\bar{j}\quad\text{and}\quad(C,D)=(k,l)\text{ or }\bar{k}\cap\bar{l}.$$

(SLLS)

$$\begin{aligned}
&\langle j(j{-}1,j{+}1)(i{-}1,i)(k,l)\rangle\langle j(j{-}1,j{+}1)(i{-}1,i)\;\bar{k}\cap\bar{l}\rangle = 0\,,\\
&\langle j(j{-}1,j{+}1)(i,i{+}1)(k,l)\rangle\langle j(j{-}1,j{+}1)(i{-}1,i)\;\bar{k}\cap\bar{l}\rangle = 0\,,\\
&\langle i(i{-}1,i{+}1)(j{-}1,j)(k,l)\rangle\langle j(j{-}1,j{+}1)(i{-}1,i)\;\bar{k}\cap\bar{l}\rangle = 0\,,\\
&\langle i(i{-}1,i{+}1)(j,j{+}1)(k,l)\rangle\langle j(j{-}1,j{+}1)(i{-}1,i)\;\bar{k}\cap\bar{l}\rangle = 0\,.
\end{aligned}$$

Dennen, Spradlin, AV

$$\langle i\bar{j}\rangle\langle\bar{i}j\rangle = 0 \quad\text{and}\quad \langle k\bar{l}\rangle\langle\bar{k}l\rangle = 0$$

($\mathbf{S^2}$LLS)

$$\begin{aligned}
&\langle i\;i{+}1\;j{-}1\;j\rangle\langle j{-}1\;j\;k\;l\rangle\langle k\;l\;i\;i{+}1\rangle\langle j{-}1\;j\;\bar{k}\cap\bar{l}\rangle\langle\bar{k}\cap\bar{l}\;i\;i{+}1\rangle = 0\,, && \langle\bar{i}\cap(i,j{-}1,j)\;\bar{l}\cap(k,k{+}1,l)\rangle = 0\,,\\
&\langle i{-}1\;i\;j{-}1\;j\rangle\langle j{-}1\;j\;k\;l\rangle\langle k\;l\;i{-}1\;i\rangle\langle j{-}1\;j\;\bar{k}\cap\bar{l}\rangle\langle\bar{k}\cap\bar{l}\;i{-}1\;i\rangle = 0\,, && \langle\bar{i}\cap(i,j,j{+}1)\;\bar{l}\cap(k,k{+}1,l)\rangle = 0\,,\\
&\langle i\;i{+}1\;j\;j{+}1\rangle\langle j\;j{+}1\;k\;l\rangle\langle k\;l\;i\;i{+}1\rangle\langle j{-}1\;j\;\bar{k}\cap\bar{l}\rangle\langle\bar{k}\cap\bar{l}\;i{-}1\;i\rangle = 0\,, && \langle\bar{i}\cap(i,j{-}1,j)\;\bar{l}\cap(k{-}1,k,l)\rangle = 0\,,\\
&\langle i{-}1\;i\;j\;j{+}1\rangle\langle j\;j{+}1\;k\;l\rangle\langle k\;l\;i{-}1\;i\rangle\langle j{-}1\;j\;\bar{k}\cap\bar{l}\rangle\langle\bar{k}\cap\bar{l}\;i{-}1\;i\rangle = 0\,, && \langle\bar{i}\cap(i,j,j{+}1)\;\bar{l}\cap(k{-}1,k,l)\rangle = 0\,,
\end{aligned}$$

It would be very difficult to solve Landau equations w/o momentum twistors!

So far we have produced

a long list of Landau singularities for

one and two-loop N=4 SYM integrals.

Plan

- Review: Landau equations (ELOP)
- Landau singularities for one and two-loop integrals in N=4 Yang-Mills

→Landau singularities and amplitudes: physical and spurious singularities

- Refined analysis using Amplituhedron
- Singularities from Amplituhedron: geometric algorithm
- Conclusions and open questions

Amplitudes: Symbol and Singularities

- Many of the simplest (and hence best understood) amplitudes can be expressed in terms of a class of generalized polylogs defined by iterated integrals

$$Li_k(z) = \int_0^z Li_{k-1}(t)\, d\log t \qquad Li_1(z) = -\log(1-z)$$

$$R_6^{(2)} = \sum_{\text{cyclic}} \mathrm{Li}_4\left(-\frac{\langle 1234\rangle\langle 2356\rangle}{\langle 1236\rangle\langle 2345\rangle}\right) - \frac{1}{4}\mathrm{Li}_4\left(-\frac{\langle 1246\rangle\langle 1345\rangle}{\langle 1234\rangle\langle 1456\rangle}\right)$$

$+$ products of $\mathrm{Li}_k(-x)$ functions of lower weight

Goncharov, Spradlin, Vergu, AV

- Much of the information about the analytic structure of such function is captured in symbol.

Symbol of Transcendental Function

Goncharov, Spradlin, Vergu, AV

$$T_k \to S(T_k) = R_1 \otimes \cdots \otimes R_k$$

Symbol is an element of the k-fold tensor product of the multiplicative group of rational functions.

$$log(R) \to R$$

$$Li_2(R) \to -(1-R) \otimes R$$

$$dT_k = \sum_i T^i_{k-1} d\log R_i \to S(T_k) = \sum_i S(T^i_{k-1}) \otimes R_i$$

$$R_6^{(2)} = \sum_{\text{cyclic}} \text{Li}_4\left(-\frac{\langle 1234\rangle\langle 2356\rangle}{\langle 1236\rangle\langle 2345\rangle}\right) - \frac{1}{4}\text{Li}_4\left(-\frac{\langle 1246\rangle\langle 1345\rangle}{\langle 1234\rangle\langle 1456\rangle}\right)$$

$+$ products of $\text{Li}_k(-x)$ functions of lower weight

$$\langle 1256\rangle \otimes \langle 1346\rangle \otimes \langle 1246\rangle \otimes \langle 1456\rangle + \cdots$$ 7272 terms

Symbol and Singularities

- Much of the information about the analytic structure of such function is captured in symbol.
- For amplitudes of generalized polylogarithm form there should be a close connection between Landau singularities and symbol alphabet of the amplitude.
- We expect that the symbol entries appearing in any amplitude should be such that their zeros specify values of the external momenta where solutions of the Landau equations exist.

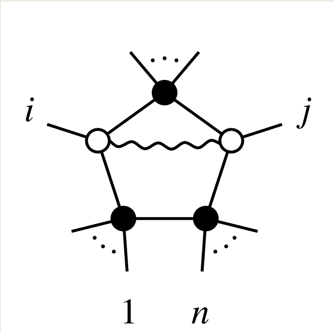
Maldacena, Simons-Duffin, Zhiboedov 2015

Abreu, Britto, Duhr, Gardi, Gronqvist 2014

One-loop n-point MHV in N=4 SYM

Bern, Dixon, Dunbar, Kosower

Arkani-Hamed, Bourjaily, Cachazo, Trnka

$$\int_{AB}$$ 

$$= \begin{aligned} &\mathrm{Li}_2\left(1-u_{n,i-1,i,j}\right)-\mathrm{Li}_2\left(1-u_{j,n,i,j-1}\right)-\mathrm{Li}_2\left(1-u_{i,j-1,n,i-1}\right)\\ &-\mathrm{Li}_2\left(1-u_{i,j-1,n,i-1}\right)+\mathrm{Li}_2\left(1-u_{i,j-1,j,i-1}\right)\\ &+\log\left(u_{j,n,i-1,j-1}\right)\log\left(u_{n,i-1,i,j}\right)\end{aligned}$$

$$u_{i,j,k,l}=\frac{\langle i\, i{+}1\, j\, j{+}1\rangle\langle k\, k{+}1\, l\, l{+}1\rangle}{\langle l\, l{+}1\, j\, j{+}1\rangle\langle k\, k{+}1 i\, i{+}1\rangle}=\frac{x_{ij}^2 x_{kl}^2}{x_{lj}^2 x_{ki}^2}$$

Compute the Symbol

First Entry

$$\langle i{-}1\ i\ j{-}1\ j\rangle, \quad \langle i{-}1\ i\ j\ j{+}1\rangle, \quad \langle i{-}1\ i\ n\ 1\rangle, \quad \langle i\ i{+}1\ j{-}1\ j\rangle$$
$$\langle i\ i{+}1\ j\ j{+}1\rangle, \quad \langle i\ i{+}1\ n\ 1\rangle, \quad \langle j{-}1\ j\ n\ 1\rangle, \quad \langle j\ j{+}1\ n\ 1\rangle$$

Second Entry

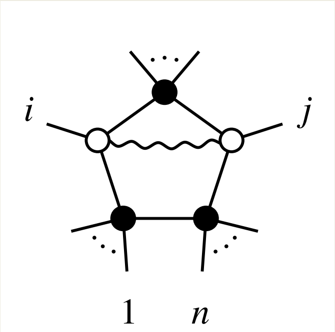
$\langle i{-}1\, i\, n\, 1\rangle$, $\langle i\, i{+}1\, n\, 1\rangle$, $\langle j{-}1\, j\, n\, 1\rangle$ and $\langle j\, j{+}1\, n\, 1\rangle$

$$\langle i\bar{j}\rangle\langle\bar{i}j\rangle$$

Summary: $\langle a\,\bar{b}\rangle=0$ or $\langle a\, a{+}1\, b\, b{+}1\rangle=0$

One-loop n-point MHV in N=4 SYM

Dennen, Spradlin, AV



$$= \frac{\langle AB\,\bar{i}\cap\bar{j}\rangle\langle i\,j\,n\,1\rangle}{\langle AB\,i{-}1\,i\rangle\langle AB\,i\,i{+}1\rangle\langle AB\,j{-}1\,j\rangle\langle AB\,j\,j{+}1\rangle\langle AB\,n\,1\rangle}$$

(LLS) $$\langle i\,j\,n\,1\rangle\langle n\,1\,\bar{i}\cap\bar{j}\rangle = 0$$ Prefactor

(SLLS)

$$\begin{aligned}
&\langle j(j{-}1,j{+}1)(i,i{+}1)(n,1)\rangle = 0\,,\\
&\langle j(j{-}1,j{+}1)(i{-}1,i)(n,1)\rangle = 0\,,\\
&\langle i(i{-}1,i{+}1)(j,j{+}1)(n,1)\rangle = 0\,,\\
&\langle i(i{-}1,i{+}1)(j{-}1,j)(n,1)\rangle = 0\,,\\
&\langle \bar{i}j\rangle\langle i\bar{j}\rangle = 0\,.
\end{aligned}$$

Second entries of the symbol

($\mathbf{S^2}$LLS)

$$\begin{aligned}
&\langle i{-}1\;i\;j{-}1\;j\rangle\langle j{-}1\;j\;n\;1\rangle\langle n\;1\;i{-}1\;i\rangle = 0\,,\\
&\langle i\;i{+}1\;j{-}1\;j\rangle\langle j{-}1\;j\;n\;1\rangle\langle n\;1\;i\;i{+}1\rangle = 0\,,\\
&\langle i{-}1\;i\;j\;j{+}1\rangle\langle j\;j{+}1\;n\;1\rangle\langle n\;1\;i{-}1\;i\rangle = 0\,,\\
&\langle i\;i{+}1\;j\;j{+}1\rangle\langle j\;j{+}1\;n\;1\rangle\langle n\;1\;i\;i{+}1\rangle = 0\,.
\end{aligned}$$

First/Second entries of the symbol

Two-loop n-point MHV in N=4 SYM

- Explicit analytic results for the chiral double pentagon have only been obtained in n=6.

Drummond, Henn

- Symbol of two-loop n-point MHV amplitude

Caron-Huot

$$\langle a\,\bar{b}\rangle \qquad \langle a\,a{+}1\,\bar{b}\cap\bar{c}\rangle$$

$$\langle a\,b\,c\,c{+}1\rangle \qquad \langle a\,(a{-}1\,a{+}1)(b\,b{+}1)(c\,c{+}1)\rangle$$

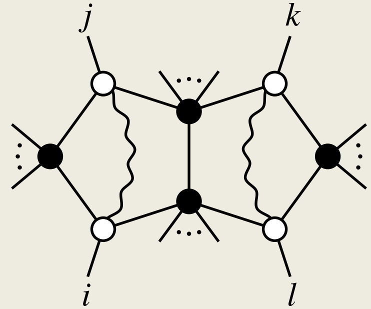
- It can be that individual chiral double pentagon integrals have an even larger symbol alphabet, with nontrivial cancelation in the sum which gives the amplitude.

- All of symbol entries are on the list of Landau singularities.

Dennen, Spradlin, AV

Two-loop n-point MHV in N=4 SYM

$$\frac{\mathcal{A}^{2-\text{loop}}_{\text{MHV}}}{\mathcal{A}^{\text{tree}}_{\text{MHV}}} = \int_{AB}\int_{CD}\frac{1}{2}\sum_{i<j<k<l<i}$$



Arkani-Hamed, Bourjaily, Cachazo, Trnka

$$\frac{\langle i\, j\, k\, l\rangle}{\langle ABCD\rangle}\frac{\langle AB\,\bar{i}\cap\bar{j}\rangle}{\langle AB\, i{-}1\, i\rangle\langle AB\, i\, i{+}1\rangle\langle AB\, j{-}1\, j\rangle\langle AB\, j\, j{+}1\rangle}\frac{\langle CD\,\bar{k}\cap\bar{l}\rangle}{\langle CD\, k{-}1\, k\rangle\langle CD\, k\, k{+}1\rangle\langle CD\, l{-}1\, l\rangle\langle CD\, l\, l{+}1\rangle}$$

(LLS)

$$\langle i\ j\ k\ l\rangle\langle i\ j\ \bar{k}\cap\bar{l}\rangle\langle\bar{i}\cap\bar{j}\ k\ l\rangle\langle\bar{i}\cap\bar{j}\ \bar{k}\cap\bar{l}\rangle = 0 \qquad (A,B)=(i,j)\text{ or }\bar{i}\cap\bar{j}\quad\text{and}\quad(C,D)=(k,l)\text{ or }\bar{k}\cap\bar{l}\,.$$

(SLLS)

$$\begin{aligned}
&\langle j(j{-}1,j{+}1)(i{-}1,i)(k,l)\rangle\langle j(j{-}1,j{+}1)(i{-}1,i)\ \bar{k}\cap\bar{l}\rangle = 0\,,\\
&\langle j(j{-}1,j{+}1)(i,i{+}1)(k,l)\rangle\langle j(j{-}1,j{+}1)(i{-}1,i)\ \bar{k}\cap\bar{l}\rangle = 0\,,\\
&\langle i(i{-}1,i{+}1)(j{-}1,j)(k,l)\rangle\langle j(j{-}1,j{+}1)(i{-}1,i)\ \bar{k}\cap\bar{l}\rangle = 0\,,\\
&\langle i(i{-}1,i{+}1)(j,j{+}1)(k,l)\rangle\langle j(j{-}1,j{+}1)(i{-}1,i)\ \bar{k}\cap\bar{l}\rangle = 0\,.
\end{aligned}$$

Dennen, Spradlin, AV

$$\langle i\bar{j}\rangle\langle\bar{i}j\rangle = 0\quad\text{and}\quad\langle k\bar{l}\rangle\langle\bar{k}l\rangle = 0$$

(S²LLS)

$$\begin{aligned}
&\langle i\ i{+}1\ j{-}1\ j\rangle\langle j{-}1\ j\ k\ l\rangle\langle k\ l\ i\ i{+}1\rangle\langle j{-}1\ j\ \bar{k}\cap\bar{l}\rangle\langle\bar{k}\cap\bar{l}\ i\ i{+}1\rangle = 0\,,\\
&\langle i{-}1\ i\ j{-}1\ j\rangle\langle j{-}1\ j\ k\ l\rangle\langle k\ l\ i{-}1\ i\rangle\langle j{-}1\ j\ \bar{k}\cap\bar{l}\rangle\langle\bar{k}\cap\bar{l}\ i{-}1\ i\rangle = 0\,,\\
&\langle i\ i{+}1\ j\ j{+}1\rangle\langle j\ j{+}1\ k\ l\rangle\langle k\ l\ i\ i{+}1\rangle\langle j{-}1\ j\ \bar{k}\cap\bar{l}\rangle\langle\bar{k}\cap\bar{l}\ i{-}1\ i\rangle = 0\,,\\
&\langle i{-}1\ i\ j\ j{+}1\rangle\langle j\ j{+}1\ k\ l\rangle\langle k\ l\ i{-}1\ i\rangle\langle j{-}1\ j\ \bar{k}\cap\bar{l}\rangle\langle\bar{k}\cap\bar{l}\ i{-}1\ i\rangle = 0\,,
\end{aligned}$$

$$\begin{aligned}
&\langle\bar{i}\cap(i,j{-}1,j)\ \bar{l}\cap(k,k{+}1,l)\rangle = 0\,,\\
&\langle\bar{i}\cap(i,j,j{+}1)\ \bar{l}\cap(k,k{+}1,l)\rangle = 0\,,\\
&\langle\bar{i}\cap(i,j{-}1,j)\ \bar{l}\cap(k{-}1,k,l)\rangle = 0\,,\\
&\langle\bar{i}\cap(i,j,j{+}1)\ \bar{l}\cap(k{-}1,k,l)\rangle = 0\,,
\end{aligned}$$

Landau Singularities and Symbology

- All symbol entries are Landau singularities.
- Can we make a stronger statement? Why various other Landau singlularities don't appear in the symbol?
- SSLLS involve more complicated four-brackets than those which appear in amplitudes, but they are similar to cluster A-coordinates for the Grassmannian cluster algebra that it relevant to planar SYM.

$$\langle \bar{i} \cap (i, j{-}1, j)\ \bar{l} \cap (k, k{+}1, l) \rangle = 0$$
$$\langle \bar{i} \cap (i, j, j{+}1)\ \bar{l} \cap (k, k{+}1, l) \rangle = 0$$
$$\langle \bar{i} \cap (i, j{-}1, j)\ \bar{l} \cap (k{-}1, k, l) \rangle = 0$$
$$\langle \bar{i} \cap (i, j, j{+}1)\ \bar{l} \cap (k{-}1, k, l) \rangle = 0$$

- We can try to explore spurious singularities using cluster algebras.
- All evidence to date says that for the simplest amplitudes in planar N=4 Yang-Mills symbol entries are cluster coordinates on Gr(4,n).

Goncharov, Spradlin, Vergu; Golden, Paulos, Parker, Scherlis, AV

A_2

$$a_1 \longrightarrow a_2$$

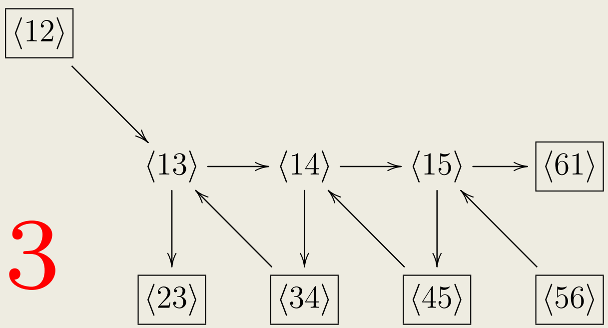
$$a_1,\ a_2,\ a_3 = \frac{1+a_2}{a_1},\ a_4 = \frac{1+a_1+a_2}{a_1 a_2},\ a_5 = \frac{1+a_1}{a_2}$$

$Gr(4,n)$

$$\begin{tikzcd}
\boxed{\langle 1234\rangle} \arrow[dr] & & & & & \\
& \langle 1235\rangle \arrow[r] \arrow[d] & \langle 1236\rangle \arrow[r] \arrow[d] & \cdots \arrow[r] & \langle 123\,n-1\rangle \arrow[r] \arrow[d] & \boxed{\langle 123n\rangle} \\
& \langle 1245\rangle \arrow[r] \arrow[d] & \langle 1256\rangle \arrow[r] \arrow[d] \arrow[ul] & \cdots \arrow[r] \arrow[ul] & \langle 12\,n-2\,n-1\rangle \arrow[r] \arrow[d] \arrow[ul] & \boxed{\langle 12\,n-1\,n\rangle} \arrow[ul] \\
& \langle 1345\rangle \arrow[r] \arrow[d] & \langle 1456\rangle \arrow[r] \arrow[d] \arrow[ul] & \cdots \arrow[r] \arrow[ul] & \langle 1\,n-3\,n-2\,n-1\rangle \arrow[r] \arrow[d] \arrow[ul] & \boxed{\langle 1\,n-2\,n-1\,n\rangle} \arrow[ul] \\
& \boxed{\langle 2345\rangle} & \boxed{\langle 3456\rangle} \arrow[ul] & \cdots \arrow[ul] & \boxed{\langle n-4\,n-3\,n-2\,n-1\rangle} \arrow[ul] & \boxed{\langle n-3\,n-2\,n-1\,n\rangle} \arrow[ul]
\end{tikzcd}$$

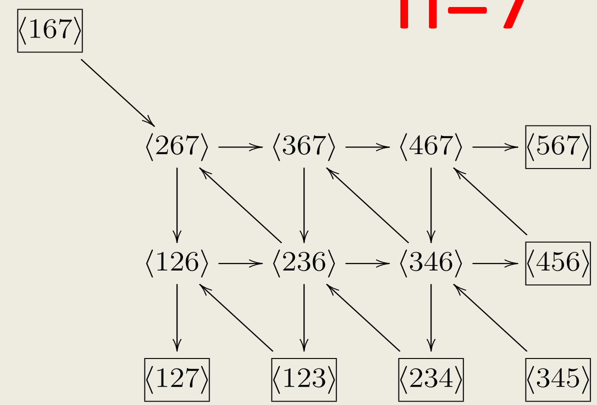
n=6

A_3



n=7

E_6



- Cluster algebras structure has been used for advancing computations of multi-loop N=4 Yang-Mills amplitudes.
- Exploring cluster algebras at more then 8 points becomes very hard, and it will be interesting to explore the connection in details.
- How can we get rid of spurious singularities?
- Instead let us turn to ……

TIME

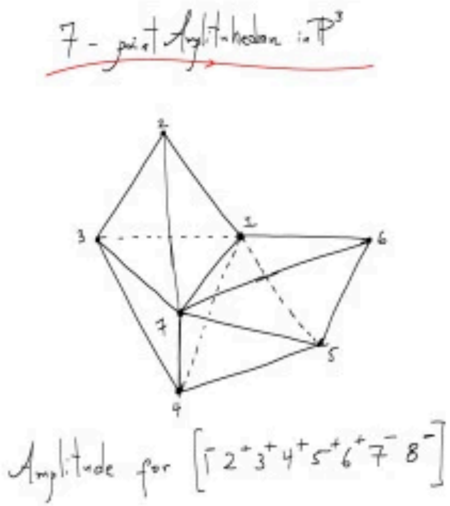
The 25 Best Inventions of the Year 2013

Interesting

The Amplituhedron

By TIME Staff | Nov. 13, 2013

Physicists at the Institute for Advanced Study in Princeton, N.J., recently found a major shortcut for predicting subatomic-particle collisions. The new method **represents probabilities as pyramid-like structures,** then combines the pyramids into one elegant gemstone-like structure called an amplituhedron, thereby massively simplifying the task of calculating particle interactions. Ultimately the amplituhedron could lead to the long-sought quantum theory of gravity.



NIMA

Plan

- Review: Landau equations (ELOP)
- Landau singularities for one and two-loop integrals in N=4 Yang-Mills
- Landau singularities and amplitudes: physical and spurious singularities

→Refined analysis using Amplituhedron

- Singularities from Amplituhedron: geometric algorithm
- Conclusions and open questions

Amplituhedron

Arkani-Hamed, Trnka 2013

- The integrand of an L-loop MHV amplitude is a rational function of the momentum twistors Z_i (external kinematics) and L loop momenta (each of which corresponds to some line $\mathcal{L}^{(\ell)}$ in $\mathbb{P}^3$)
- Amplituhedron provides a construction for amplitudes when Z_i take values in the positive domain $G_+(4,n)$
- Each line $\mathcal{L}^{(\ell)}$ in $\mathbb{P}^3$ maybe characterized by a pair of points $\mathcal{L}_1^{(\ell)}, \mathcal{L}_2^{(\ell)}$
- In the MHV amplituhedron, a pair of points specifying each $\mathcal{L}^{(\ell)}$ may be expressed in the Z_i basis via D-matrix:

$$\mathcal{L}_\alpha^{(\ell)I} = \sum_{i=1}^{n} D_{\alpha i}^{(\ell)} Z_i^I, \qquad \alpha = 1,2$$

Amplituhedron

Arkani-Hamed, Trnka

- Inside the amplituhedron, D-matrices are in $G_+(2,n)$

$$\langle \mathcal{L}^{(\ell)}\, i\, j \rangle > 0 \text{ for } i < j \text{ and all } \ell, \;\text{ and}$$
$$\langle \mathcal{L}^{(\ell_1)}\, \mathcal{L}^{(\ell_2)} \rangle > 0 \text{ for all } \ell_1, \ell_2.$$

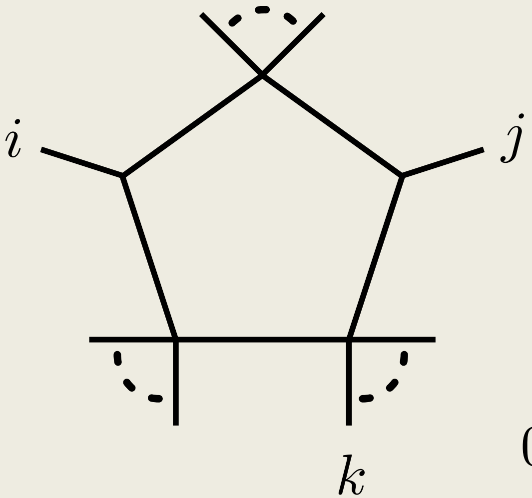
- Boundaries occur when one or more of these equalities approach zero.
- The integrand of an MHV amplitude is a canonical form defined by having logarithmic singularities only on the boundary of the amplituhedron.

How can we use Amplituhedron to eliminate spurious singularities?

Use positivity constraints in the amplituhedron!

Example: Spurious Three-Mass Box

Dennen, Prlina, Spradlin, Stanojevic, AV



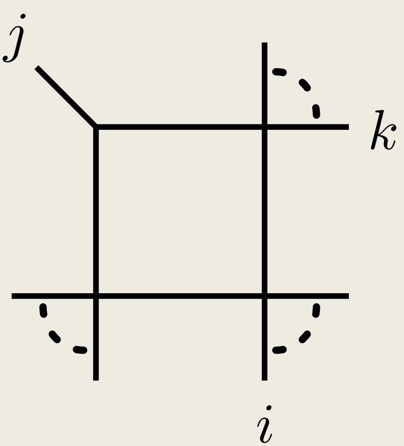
$$\langle j\,(j{-}1\,j{+}1)(i\,i{+}1)(k\,k{+}1)\rangle = 0$$

It arises from the cut conditions:

$$0 = \langle \mathcal{L}\, i\, i{+}1\rangle = \langle \mathcal{L}\, j{-}1\, j\rangle = \langle \mathcal{L}\, j\, j{+}1\rangle = \langle \mathcal{L}\, k\, k{+}1\rangle$$

These are three-mass box type and have solutions

$$\mathcal{L} = (j\, i\, i{+}1) \cap (j\, k\, k{+}1) \text{ or } \mathcal{L} = (\bar{j} \cap (i\, i{+}1), \bar{j} \cap (k\, k{+}1))$$



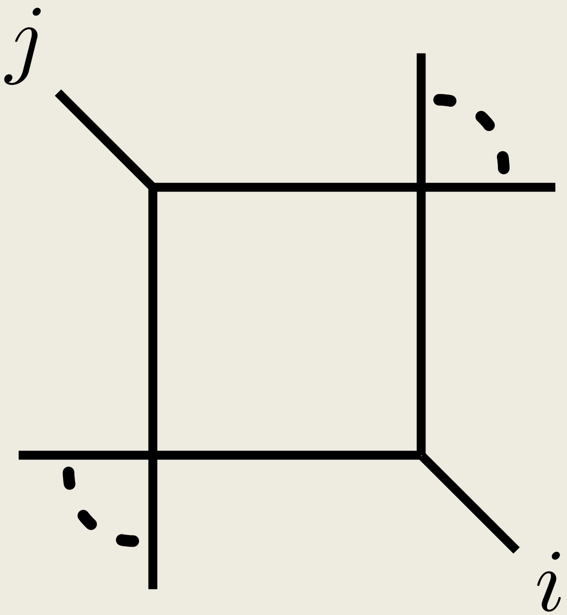
Solutions can be represented by D-matrix.

$$D = \begin{array}{c} \begin{array}{ccc} i & i{+}1 & j \end{array} \\ \begin{pmatrix} 0 & 0 & 1 \\ \langle i{+}1\, j\, k\, k{+}1\rangle & -\langle i\, j\, k\, k{+}1\rangle & 0 \end{pmatrix} \end{array}$$

D matrices are not positive definite when Z are positive!

$$D = \begin{array}{c} \begin{array}{cccc} i & i{+}1 & k & k{+}1 \end{array} \\ \begin{pmatrix} \langle i{+}1\,\bar{j}\rangle & -\langle i\,\bar{j}\rangle & 0 & 0 \\ 0 & 0 & -\langle \bar{j}\, k+1\rangle & \langle \bar{j}\, k\rangle \end{pmatrix} \end{array}$$

Compare: Two-Mass Easy Box



$$\langle i\, \bar{j} \rangle \langle \bar{i}\, j \rangle = 0$$

It arises from the cut conditions:

$$\langle \mathcal{L}\, i{-}1\, i \rangle = \langle \mathcal{L}\, i\, i{+}1 \rangle = \langle \mathcal{L}\, j{-}1\, j \rangle = \langle \mathcal{L}\, j\, j{+}1 \rangle = 0$$

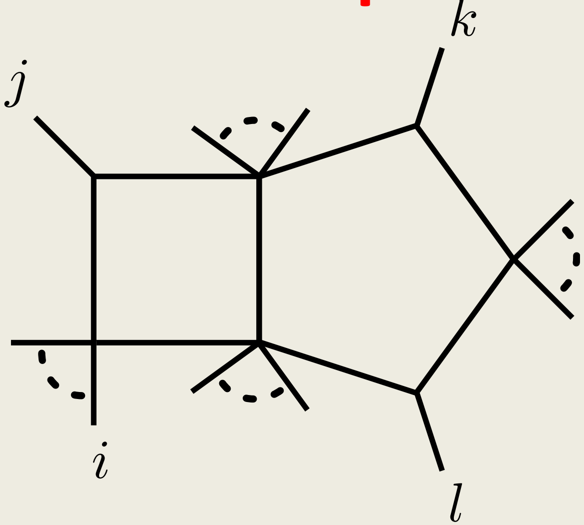
Cut conditions have a solution:

$$\mathcal{L} = (i\, j)$$

Positive D-matrix!!

$$D = \begin{array}{c} \begin{matrix} i & j \end{matrix} \\ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{array}$$

Example: two-loop double-pentagon



$$\langle j\,(j{-}1\,j{+}1)(i{-}1\,i)\,(k\,l)\rangle\langle j\,(j{-}1\,j{+}1)(i{-}1\,i)\,\bar{k}\cap\bar{l}\rangle = 0$$

It arises from the cut conditions:

$$\langle \mathcal{L}^{(1)}\,i{-}1\,i\rangle = \langle \mathcal{L}^{(1)}\,j{-}1\,j\rangle = \langle \mathcal{L}^{(1)}\,j\,j{+}1\rangle = \langle \mathcal{L}^{(1)}\,\mathcal{L}^{(2)}\rangle = 0\,,$$
$$\langle \mathcal{L}^{(2)}\,k{-}1\,k\rangle = \langle \mathcal{L}^{(2)}\,k\,k{+}1\rangle = \langle \mathcal{L}^{(2)}\,l{-}1\,l\rangle = \langle \mathcal{L}^{(2)}\,l\,l{+}1\rangle = 0$$

The last line can be solved by $\mathcal{L}^{(2)} = (k\,l)$ or $\mathcal{L}^{(2)} = \bar{k}\cap\bar{l},$

Taking the first solution, solve the first line

$$\mathcal{L}^{(1)} = (j\,i{-}1\,i)\cap(j\,k\,l) = (Z_j, Z_{i-1}\langle i\,j\,k\,l\rangle - Z_i\langle i{-}1\,j\,k\,l\rangle) \text{ or}$$
$$\mathcal{L}^{(1)} = \Big((i{-}1\,i)\cap\bar{j}, (k\,l)\cap\bar{j}\Big) = \Big(Z_{i-1}\langle i\,\bar{j}\rangle - Z_i\langle i{-}1\,\bar{j}\rangle, Z_k\langle l\bar{j}\rangle - Z_l\langle k\,\bar{j}\rangle\Big)$$

Minors do not have uniform sign!

$$\begin{pmatrix} D^{(1)} \\ D^{(2)} \end{pmatrix} = \begin{array}{c} \begin{array}{ccccc} i{-}1 & i & j & k & l \end{array} \\ \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ \langle i\,j\,k\,l\rangle & -\langle i{-}1\,j\,k\,l\rangle & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

$$\begin{pmatrix} D^{(1)} \\ D^{(2)} \end{pmatrix} = \begin{array}{c} \begin{array}{cccc} i{-}1 & i & k & l \end{array} \\ \begin{pmatrix} \langle i\,\bar{j}\rangle & -\langle i{-}1\,\bar{j}\rangle & 0 & 0 \\ 0 & 0 & \langle l\bar{j}\rangle & -\langle k\,\bar{j}\rangle \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

Summary so far

Dennen, Prlina, Spradlin, Stanojevic, AV

- First, consider a representation of an amplitude as a sum over particular integrals. Find Landau singularities of a generic term in the sum. These tell us potential singularities.
- Second, for each potential singularity check whether the corresponding on-shell conditions have a non-zero intersection with the (closure of) the amplituhedron. If the answer is no, then the singularity must be spurious.
- This approach is straightforward but inefficient.
- The most significant drawback of the approach is that it relies on having an explicit representation of an integrand in terms on local Feynman integrals.

An Amplituhedrony Approach

Dennen, Prlina, Spradlin, Stanojevic, AV

- Instead of using Feynman diagrams to generate sets of cut conditions that we need to check one by one, we can ask amplituhedron itself directly to identify all potentially "valid" sets of cut conditions that are possibly relevant to the singularities of an amplitude.
- For each valid set of cut conditions, solve Landau equations and find the corresponding singularity.

Plan

- Review: Landau equations (ELOP)
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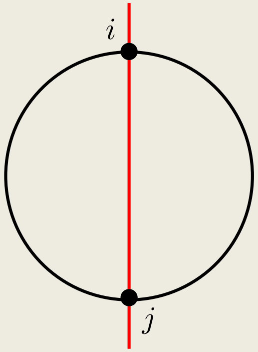
→ Singularities from Amplituhedron: geometric algorithm

- Conclusions and open questions

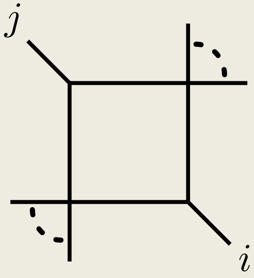
One-loop MHV



- The maximum codimension boundaries for the one-loop MHV amplituhedron occur when

$$\mathcal{L} = (i\, j)$$

- On this boundary four cut conditions of the two-mass easy type are satisfied



$$\langle \mathcal{L}\, i{-}1\, i \rangle = \langle \mathcal{L}\, i\, i{+}1 \rangle = \langle \mathcal{L}\, j{-}1\, j \rangle = \langle \mathcal{L}\, j\, j{+}1 \rangle = 0$$

- A leading solution to the Landau equations for two-mass easy box exists only if

$$\langle i\, \bar{j} \rangle \langle \bar{i}\, j \rangle = 0$$

One-loop MHV Amplitude

- Subleading Landau equations are obtained by relaxing one of the four on-shell conditions. This leads to cuts of two-mass triangle type, which don't give anything.
- At sub-subleading order we reach cuts of bubble type. For instance, we encounter

$$\langle i{-}1\, i\, j{-}1\, j \rangle = 0$$

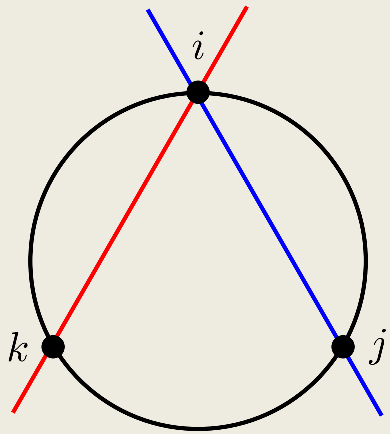
- Altogether, we all physical branch points occur on loci of the form

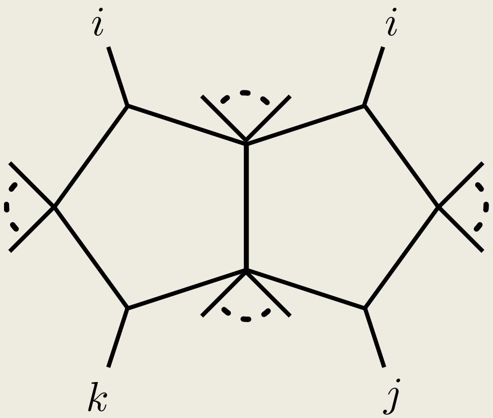
$$\langle a\, \bar{b} \rangle = 0 \text{ or } \langle a\, a{+}1\, b\, b{+}1 \rangle = 0$$

- These are precisely the singularities of the one-loop MHV amplitudes (symbol alphabet)!

Two-Loop MHV Amplitudes: configurations of positive lines

Maximum codimension boundary

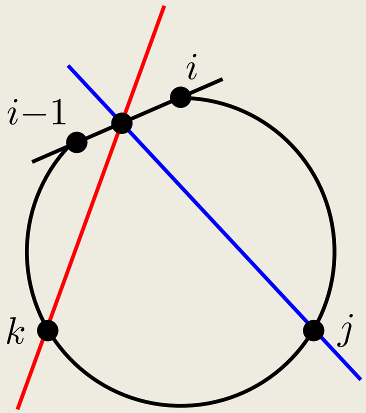




On this boundary the following nine cut conditions are satisfied

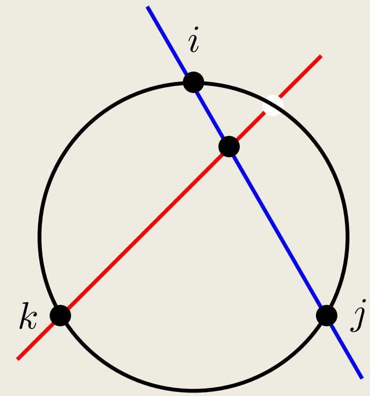
$$\langle \mathcal{L}^{(1)}\, i{-}1\, i\rangle = \langle \mathcal{L}^{(1)}\, i\, i{+}1\rangle = \langle \mathcal{L}^{(2)}\, i{-}1\, i\rangle = \langle \mathcal{L}^{(2)}\, i\, i{+}1\rangle = 0\,,$$
$$\langle \mathcal{L}^{(1)}\, j{-}1\, j\rangle = \langle \mathcal{L}^{(1)}\, j\, j{+}1\rangle = \langle \mathcal{L}^{(2)}\, k{-}1\, k\rangle = \langle \mathcal{L}^{(2)}\, k\, k{+}1\rangle = 0\,,$$
$$\langle \mathcal{L}^{(1)}\, \mathcal{L}^{(2)}\rangle = 0\,.$$

Two-Loop MHV Amplitudes: two valid double relaxations



(a)

$\langle \mathcal{L}^{(2)}\, i\, i{+}1\rangle, \langle \mathcal{L}^{(1)}\, i\, i{+}1\rangle \neq 0$



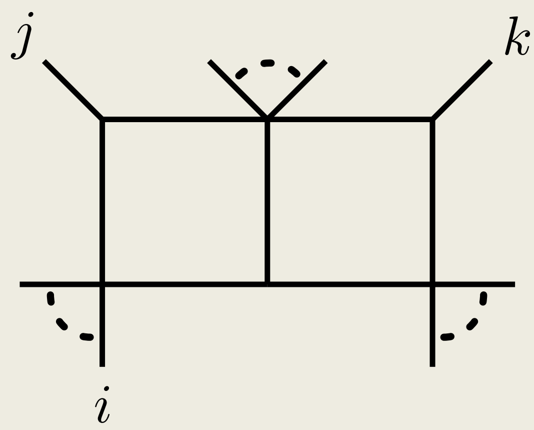
(b)

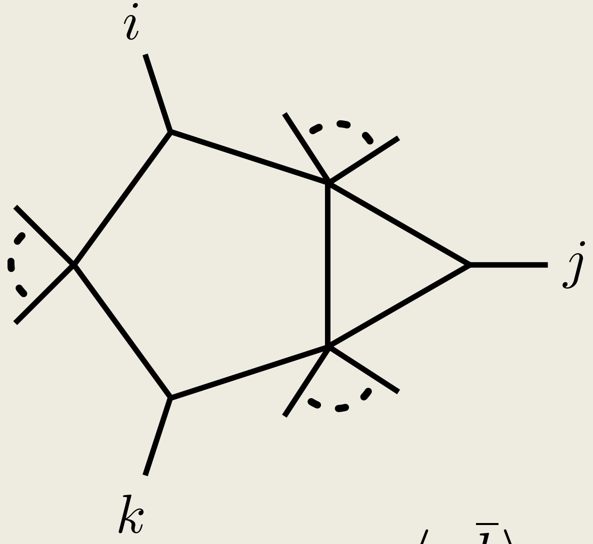
$\langle \mathcal{L}^{(2)}\, i{-}1\, i\rangle, \langle \mathcal{L}^{(2)}\, i\, i{+}1\rangle \neq 0$

$$\begin{pmatrix} D^{(1)} \\ D^{(2)} \end{pmatrix} = \begin{array}{c} \begin{array}{cccc} i & i{+}1 & j & k \end{array} \\ \begin{pmatrix} \alpha & 1-\alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \alpha & 1-\alpha & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

$$\begin{pmatrix} D^{(1)} \\ D^{(2)} \end{pmatrix} = \begin{array}{c} \begin{array}{ccc} i & j & k \end{array} \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \alpha & 1-\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{array}$$

Two-Loop MHV Amplitudes: Landau diagrams for each valid relaxation





Solving Landau equations give singularities

$$\langle a\,\bar{b}\rangle = 0$$

$$\langle a\,b\,c\,c{+}1\rangle = 0$$

$$\langle a\,a{+}1\,\bar{b}\cap\bar{c}\rangle = 0$$

$$\langle a\,(a{-}1\,a{+}1)(b\,b{+}1)(c\,c{+}1)\rangle = 0$$

These are precisely the singularities of the two-loop MHV amplitudes (symbol alphabet)!

Summary: geometric algorithm

- Input: list of the maximal codimension boundaries of the amplituhedron. These are known for MHV.
- Step 1: Identify the list of all cut conditions satisfied on the given boundary, and consider all lower codimension boundaries by relaxing various subsets. Eliminate those which do not overlap with the closure of the amplituhedron.
- Step 2: For each valid set of cut conditions, solve the corresponding Landau equations to determine the location of singularities.
- Output: A list of the loci in external kinematics space where the given amplitude has branch points.

Dennen, Prlina, Spradlin, Stanojevic, AV

Conclusion

- We proposed a geometric algorithm to determine singularities of amplitudes in N=4 SYM from the amplituhedron.
- We applied the algorithm to the one- and two-loop MHV amplitudes.
- This is a step towards translating integrands directly to amplitudes.
- Many questions remain:

--generalizations to other cases

--relation between cluster structure and Landau singularities


HAPPY
BIRTHDAY