HYPERKÄHLER METRICS ON A 4-MANIFOLD WITH BOUNDARY

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Informal summary of https://arxiv.org/abs/1603.08170, joint work with Joel Fine and Jason Lotay

1. Dedication

The work described here is not so close to my work in Roger's research group where I started as a D. Phil student in the early 1980s, though much of what I learned at that time—and much more that I assimilated by osmosis—has inspired me, at a deeper level, throughout my mathematical career. I count it an enormous privilege to have been Roger's student, to have worked with him, and to have benefitted from the many ideas revealed in a decade of Friday meetings at Oxford.

I am also very pleased to have learned about geometric quantization in my early years at Oxford from Nick Woodhouse, his book, and his papers. Discussions of positive/negative frequency decompositions seem to me to have been quite frequent in Roger's group at that time and I am glad to have been educated about the physical importance and geometric context of these ideas at an early stage.

The work described here sets positive/negative fequency decompositions in a geometric context very close to the nonlinear graviton construction.

Roger and Nick: my warmest wishes for the future!

2. Disclaimer

This exegesis is based on the preprint https://arxiv.org/abs/1603.08170 by Joel Fine, Jason Lotay and myself. Any errors, sloppiness, or failures of referencing here are mine and mine alone. The reader is referred to the above for full details, references and greater precision.

3. HyperKähler metrics and HyperKähler triples

We are concerned with the 'euclidean signature' version of Roger Penrose's famous nonlinear graviton construction: hyperKähler metrics in dimension 4.

These have many equivalent definitions. For example (M, g) is hyperKähler if its holonomy is (contained in) $SU(2)$; equivalently if it is Kähler in 3 (and hence infinitely many) ways, by which we mean that there are complex structures (J_1, J_2, J_3) satisfying the quaternionic relations

$$J_1^2 = J_2^2 = J_3^2 = J_1J_2J_3 = -1 \tag{3.1}$$

and which are g-parallel (hence integrable). Equivalently again, we may work with the 2-forms $\omega_i(\cdot,\cdot) = g(J_i\cdot,\cdot)$. These are then parallel (hence closed, hence symplectic forms). The counterpart of (3.1) for the ω_i is:

$$\omega_i \wedge \omega_j = 2\mu\delta_{ij}. \tag{3.2}$$

Note that

$$\mu = \frac{1}{6}(\omega_1^2 + \omega_2^2 + \omega_3^2) \tag{3.3}$$

and (because the ω_i are non-degenerate)

$$\mu > 0. \tag{3.4}$$

It follows that the connection induced on Λ^+, the bundle of self-dual 2-forms is flat, for the ω_i give a flat basis of sections of this bundle. Hence

$$\mathrm{Ric} = 0, W^+ = 0 \text{ or } \Phi_{A'B'CD} = 0, \Lambda = 0, \Psi_{ABCD} = 0. \tag{3.5}$$

to use the notation that I learned from Roger thirty or more years ago. These vanishing curvature conditions are the ones appearing in the nonlinear graviton paper. If they are satisfied, then locally at least, Λ^+ has a trivialization by a flat orthonormal basis, and this basis defines a hyperKähler triple of 2-forms.

For an analytical study of hK metrics, it is good to work with the forms rather than the metric. Here is why this works:

3.1. **Proposition.** *On an oriented 4-manifold, suppose given a triple of 2-forms ω_i which satisfy (3.2) (from which (3.3) follows) with $\mu > 0$ at every point. Then there is a unique metric g with volume form μ, and such that the ω_i span the subspace of g-self-dual 2-forms at each point. Moreover, if $d\omega_i = 0$, then the ω_i are parallel and g is hyperKähler.*

Proof. It is well known that a positive, rank-3 subbundle of $P \subset \Lambda^2 M$ defines a conformal structure $[g]$ in such a way that P is the bundle of self-dual 2-forms with respect to $[g]$. (The notion of self-duality is conformally invariant.) We define P to be the span of the ω_i, and choose the conformal factor so that μ is the volume form of the metric. The positivity of μ means that the ω_i are independent at every point, so P is of rank 3.

Now a folklore theorem says that the connection on P, induced by the metric connection, is determined completely by $\mathrm{d}\omega_i$. In particular, if $\mathrm{d}\omega_i = 0$, then this connection is flat, the ω_i are parallel and g is hyperKähler. □

4. HyperKähler thickenings and fillings

Suppose $\Sigma^3 \subset M^4$ and M^4 has a hyperKähler triple $(\omega_1, \omega_2, \omega_3)$. Let ν be the normal of Σ, and let $\eta_i = \iota_\nu \omega_i$ be the induced framing of $T^*\Sigma$.

From the properties of ω, there is an induced metric on Σ

$$h = \sum \eta_i \otimes \eta_i \tag{4.1}$$

and we have

$$\mathrm{d}(*_\eta \eta) = 0 \tag{4.2}$$

(It is a co-closed co-framing!)

4.1. **Remark.** The same information is contained in the pull-backs to Σ of the ω_i. This gives a framing of $\Lambda^2 T^*\Sigma$ by closed 2-forms.

4.2. **Question.** Given a co-closed coframing (η_1, η_2, η_3) of a 3-manifold Σ, when does it arise from a hyperKähler triple $(\omega_1, \omega_2, \omega_3)$?

4.3. **Question.** Let M^4 be compact, smooth (and oriented) with boundary N^3. Let η be a co-closed coframing of T^*N. When does there exist a hyperKähler filling ω of η on M?

4.4. **Easier Question.** Let (M, ω) be hK with induced co-closed coframing η of N. Which nearby co-closed co-framings $\widehat{\eta}$ of η admit nearby hK fillings $\widehat{\omega}$ on M?

4.5. **Discussion.** We shall see that an appropriate analogy to keep in mind is the two-dimensional system of Cauchy–Riemann equations:

$$f_x + i f_y = 0, \tag{4.3}$$

where f is a complex-valued function of (x, y). For definiteness, suppose given a smooth, complex-valued function $\phi(\theta)$ on the unit circle. Then the Fourier expansion of ϕ may be written

$$\phi(z) = \sum_{n=-\infty}^{\infty} \phi_n z^n, \tag{4.4}$$

where $|z| = 1$. The function ϕ is real-analytic in θ if and only if the ϕ_n are rapidly decreasing, and this is necessary and sufficient for (4.4) to converge in some thin annular neighbourhood of $|z| = 1$. This convergent Laurent series gives a holomorphic function f defined near $|z| = 1$ and equal to ϕ on this circle.

For the filling problem, we need $\phi_n = 0$ for all $n < 0$ which is a *positive-frequency* condition. This is necessary and sufficient for the RHS of (4.4) to define a holomorphic function f in $|z| < 1$ whose restriction to the boundary is ϕ.

We claim that the thickening and filling problems for hyperKähler metrics (or rather, for triples) work in broadly the same way.

4.6. **Examples.** SU(2)-invariant metrics on completions of $(a, b) \times S^3$ can be readily classified using these ideas. Here S^3 is identified with $SU(2)$ and we are interested in cohomogeneity-1 hyperKähler structures. I won't write all the details, but note that there are two kinds of invariant framings of S^3: the ones which are rotated by the action and the ones that are not.

It is noteworthy that the (double cover of the) Eguchi–Hansen metric[1]

$$g_{EH} = (1 - c^4/r^4)^{-1} \mathrm{d}r^2 + r^2(1 - c^4/r^4) e_1^2 + r^2(e_2^2 + e_3^2), \quad (r > c) \tag{4.5}$$

and the Taub–NUT metric

$$g_{TN} = \frac{1}{4} \frac{r+m}{r-m} \mathrm{d}r^2 + 4m^2 \frac{r-m}{r+m} e_1^2 + (r^2 - m^2)(e_2^2 + e_3^2) \tag{4.6}$$

give rise to framings of orbits which are different (because one is fixed, the other moved by) the $SU(2)$ action, even though the metrics (for suitable choices of r, c and m) can be made to agree on a given hypersurface.

Also, the (double-cover of the) EH framing on S^3 cannot be filled, as it would locally have to be isometric to the double-cover of EH, which is singular at the bolt.

4.7. **The thickening theorem.** [LeBrun, Bryant, E. Cartan]

Let the data (Σ, η) be real-analytic. Then there exists M^4 containing Σ as a real-analytic hypersurface and a hK triple ω on M such that $\eta = \iota_\nu \omega$. Such hK 'thickenings' are unique in the sense that if (ι', M') is another, then there is an isometry $\phi : M \to M'$ (possibly after shrinking M and M', such that $\phi \circ \iota = \iota'$. (And ϕ is the identity on $\iota(\Sigma)$.)

[1] The e_i are a left-invariant orthonormal basis of 1-forms on S^3

4.8. **Remark.** In this way, the 'effective local degrees of freedom' in hK metrics are two (real-analytic) function of 3 variables. This is calculated as follows: Each η is a 1-form on a 3 manifold, hence 3 functions of 3 variables. Coclosure cuts down to 2 functions of 3 variables. There are thus 6 functions of 3 of variables. But one has to factor out by motions of Σ, 4 functions of 3 variables, leaving with just 2 functions of 3 variables.

5. HyperKähler Filling

For simplicity, suppose that M is topologically trivial so N is diffeomorphic to S^3[2]. Then we may write $\widehat{\omega} = \omega + \mathrm{d}a$, where a is a triple of 1-forms.

The perturbative problem to be solved is

$$Q(\omega + \mathrm{d}a) = 0. \tag{5.1}$$

The linearization of this is a first-order operator

$$\mathscr{D} : \Omega^1 \otimes \mathbb{R}^3 \longrightarrow C^\infty(S^2_0\mathbb{R}^3), \;\; a_{A'APQ} \longmapsto \nabla^{A'}_{(A} a_{|A'|BPQ)} \tag{5.2}$$

where a is symmetric in PQ[3].

This is part of a complex, because we haven't taken into account the gauge freedom, of which there are two types. The easier one to deal with is the freedom to add $\mathrm{d}f$ to a, where f is a triple of functions. On the other hand, there is the more geometric gauge freedom to act by diffeomorphisms equal to the identity on the boundary. These act on ω by lie derivative $= \mathrm{d}(\iota_v\omega)$ by Cartan's formula. So the combined action on the a's is:

$$a \mapsto a + \mathrm{d}f + \iota_v\omega \tag{5.3}$$

where v is a tangent vector vanishing at N and f is a triple of functions. Morally, at least, these additional 7 degrees of freedom make

$$C^\infty(M, TM) \oplus C^\infty(M, \mathbb{R}^3) \to \Omega^1 \otimes \mathbb{R}^3 \longrightarrow C^\infty(S^2_0\mathbb{R}^3) \tag{5.4}$$

into an elliptic-looking complex, except for the fact that the action by v is of order 0.

Treat the above as motivation, though it is at the technical heart of our work...by pursuing this line, we are able to prove that the moduli space of hK structures on M is a smooth infinite-dimensional manifold. For clarity, our moduli space is the quotient

$$\mathscr{M} = \mathscr{H}/\mathscr{G}_0 \tag{5.5}$$

where

$$\mathscr{H} = \{(\omega_i) : \mathrm{d}\omega_i = 0, \omega_i \wedge \omega_j = 2\mu\delta_{ij},\ \mu > 0\}. \tag{5.6}$$

and $\mathscr{G}_0$ is the group of diffeomorphisms which restrict to be the identity on N.

6. What does the tangent space look like?

We shall give a partial answer to the Question 4.4 by describing the tangent space $T_{[\omega]}\mathscr{M}$ of the moduli space, at a typical triple ω, and then explaining how to parameterize this by boundary data.

The linearization of the equations is the operator $\mathscr{D}$ mentioned above:

$$a \longmapsto S^2_0(\mathrm{d}a_i, \omega_j). \tag{6.1}$$

[2]These assumptions are removed in our preprint

[3]The symmetric PQ corresponds to having a triple of 1-forms

It turns out that gauges can be fixed by imposing

$$\mathrm{d}^* a = 0, L^*\mathrm{d}a = 0, \iota_\nu a = 0. \tag{6.2}$$

Here L is the operator $v \mapsto \mathscr{L}_v\omega = \mathrm{d}(\iota_v\omega)$. We have

$$L^*\mathrm{d}a = 0 \Leftrightarrow J_i \mathrm{d}^*\mathrm{d}a_i = 0 \text{ (summed)}. \tag{6.3}$$

In our paper, we show (modulo some care with function spaces), that

$$T_{[\omega]}\mathscr{M} = \{\mathrm{d}a : \mathscr{D}a = 0, a \text{ satisfies (6.2)}\} \tag{6.4}$$

One way to satisfy the equations is to have $\mathrm{d}_+ a_i = 0$, for then all inner products in (6.1) are completely zero. To satisfy this condition, we insist that a lie in the kernel of the $D = \mathrm{d}^* + \mathrm{d}_+$

$$D : \Omega^1(M) \longrightarrow \Omega^0(M) \oplus \Omega^2_+(M) \tag{6.5}$$

We note that $Da = 0$ implies automatically that $L^*\mathrm{d}a = 0$. For $\mathrm{d}a$ is ASD and closed, hence also coclosed. Let

$$\ker_0(D) = \{a \in \ker(D) : \iota_\nu a = 0\}. \tag{6.6}$$

Then d maps $\ker_0(D)$ bijectively on to the space of exact ASD 2-forms and hence into $T_{[\omega]}\mathscr{M}$.

The other part of $T_{[\omega]}\mathscr{M}$ comes from the infinitesimal harmonic 'wibbles'[4]. *Wibbles* are the following variations. Suppose that ω is the restriction of M from some larger manifold M'. Then for any map $\Phi : M \to M'$, a diffeomorphism onto its image, $\Phi^*(\omega)$ will be a hyperKaehler triple on M. For an infinitesimal diffeomorphism, generated by a vector field v, not necessarily 0 on N, this gives the deformation $a = \iota_v\omega$ and $\mathrm{d}a = \mathrm{d}(\iota_v\omega) = L_\omega v$. Then we need $L^*Lv = 0$ to satisfy the gauge-fixing condition. Note that

$$L^*Lv = 0, v|N = 0 \Longrightarrow Lv = 0 \tag{6.7}$$

by integration by parts, so that the true gauge group contributes nothing here. There are however plenty of infinitesimal harmonic wibbles.

6.1. **Theorem.** (With appropriate care of the function spaces) *With the above definitions,*

$$T_{[\omega]}\mathscr{M} = d\ker_0 D + L\mathscr{W} \tag{6.8}$$

where

$$\mathscr{W} = \{v \in C^\infty(M, TM) : L^*Lv = 0\}. \tag{6.9}$$

6.2. **Remark.** The sum here is very definitely not direct!

7. Boundary values and a better description of the tangent space

The nicest statements occur when the mean curvature of N (with respect to the metric background) is everywhere positive. (The sign is such that this condition holds if M is a ball in $\mathbb{R}^4$ with the flat metric!)

Let D_N be the Hodge-de Rham operator on N. Let

$$G_\lambda = \ker(D_N - \lambda) \cap \ker \mathrm{d}^*.$$

Then one can show that $G_\lambda \neq 0$ only for a discrete set of λ, $\dim G_\lambda < \infty$, and that the set of λ with $G_\lambda \neq 0$ is unbounded in both directions[5].

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[4]This terminology is not used in our preprint

[5]Working with G_λ rather than H_λ is the difference between $\ker(D)$ and $\ker_0(D)$. $\mathrm{d}^*a = 0 \Rightarrow \mathrm{d}^*_N(a|N) = 0$ if $\iota_\nu a = 0$.

The topological assumption implies $G_0 = 0$. Define

$$G_- = \bigoplus_{\lambda<0} G_\lambda, \quad H_+ = \bigoplus_{\lambda>0} H_\lambda \tag{7.1}$$

Let $\mathscr{W}_+$ be the subspace of $\mathscr{W}$ with boundary-values in H_+. Then we have:

7.1. **Theorem.** If the hyperkaehler structure ω gives the boundary positive mean curvature, then

$$T_{[\omega]}\mathscr{M} \simeq G_- \otimes \mathbb{R}^3 \oplus L\mathscr{W}_+. \tag{7.2}$$

(and L is injective on $\mathscr{W}_+$).

7.2. **Remark.** The point here is that the boundary-value of any element of $\ker_0(D)$ will lie in $G_- \oplus G_+$, but the part in G_- will completely determine the part in G_+ and is, itself, freely specifiable. In other words, the map

$$\ker_0(D) \to \Omega^0(N) \oplus \Omega^1(N) \to G_- \tag{7.3}$$

is an isomorphism. Precisely which pairs (u_-, u_+) arise as boundary-values of elements of $\ker_0(D)$ is of course a difficult question, analogous to knowing the Dirichlet-to-Neumann map for the Laplacian. In any case, we can count degrees of freedom:

Negative-frequency coclosed 1-forms on N should be counted as 2 negative-frequency functions of 3 variables. We have a triple, giving 6 negative-frequency functions. $\mathscr{W}^+$ is 4 positive-frequency functions of 3 variables. To recover Cartan's count, we should subtract the freedom to 'wibble', i.e. to move the boundary within the 4-manifold. This is 4 full-frequency functions of 3 variables. Overall we are left with 2 negative-frequency functions of 3 variables.

Thus the space of fillable deformations of a given (fillable) framing is parameterized by 2 negative-frequency functions of 3 variables, which fits well with Cartan's local count mentioned in Remark 4.8.