A positive Bondi–type mass in asymptotically de Sitter spacetimes

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Abstract

There are well-motivated definitions of total momentum for asymptotically-flat and asymptotically-anti de Sitter space-times, which also have desirable properties of positivity and rigidity (in the sense that their vanishing implies flatness). Much less has been done for asymptotically-de Sitter space-times. In this talk I consider the problem of defining mass at infinity in space-times with positive Λ (and therefore with space-like $\mathcal{I}^+$).

Based on CQG 32(2015) 205011, arXiv:1505.06637 with Laszlo Szabados, and arXiv:1505.06123.

The problem was discussed in Penrose GRG 43 (2011), 3355–3366 and has been considered by Ashtekar and coworkers in arXiv:1409.3816, 1506.06152, 1510.05593, and by Shiromizu (1994), Kator and Traschen (2002), and Chruściel and Ifsits (2016).

Background: asymptotic quantities 1

In asymptotically-flat space-times:

- one can define ADM momentum at space-like infinity and Bondi momentum at $\mathcal{I}^+$;
- and prove that both are time-like and future-pointing (given DEC) by Witten spinorial arguments;
- and prove that vanishing of the momentum implies that the space-time is flat to the future of a space-like surface (call this *rigidity*).
- There is also a positive (Bondi) mass-loss formula.

Background: asymptotic quantities 2

In asymptotically-anti de Sitter space-times:

- one can define an asymptotic set of kinematic quantities;
- and prove a 'time-like future-pointing' property by a generalised Witten argument;
- and prove rigidity: vanishing of the momentum implies that the space-time is exactly anti-de Sitter to the future of a space-like surface.
- With 'asymptotically-anti de Sitter' (as usually understood) the kinematic quantities are conserved.

Background: asymptotic quantities 3

In asymptotically-de Sitter space-times (as usually understood):

- $\mathcal{I}^+$ is space-like and its metric is free data – contrast with both previous cases;
- de Sitter space has no everywhere time-like Killing vectors, so positivity is problematic already in linear theory.

So: what should we do? Look at motivation from linear theory:

Background: motivation from linear theory

Consider linearised GR in one of these three backgrounds. Given a conserved T_{ab}, a Killing vector K^a and a space-like 3-surface S with boundary Σ obtain the basic identity

$$Q[K;\Sigma] = \int_S T_{ab} K^a dS^b = \frac{c^2}{8\pi G} \oint_\Sigma R_{abcd} f^{cd} d\Sigma^{ab}.$$

Here R_{abcd} is the linearised Riemann tensor from solving the Einstein field equations and f_{ab} is made from a valence-2 twistor:

$$f_{ab} = \omega_{AB}\epsilon_{A'B'} \text{ with } \nabla_{AA'}\omega_{BC} = i\epsilon_{A(B}K_{C)A'}$$

so

$$\nabla_{A'(A}\omega_{BC)} = 0$$

which is the valence-2 *twistor equation*, whose solutions are products of valence-1 twistors...

Background: more twistor theory

Valence-2 twistors are symmetrised products of valence-1 twistors $(\omega^A, \pi_{A'})$ which in turn satisfy

$$\nabla_{AA'}\omega_B + i\epsilon_{AB}\pi_{A'} = 0$$

$$\nabla_{AA'}\pi_{B'} + i\Lambda\epsilon_{A'B'}\omega_B = 0,$$

where $\Lambda = R/24 = \lambda/6 = \pm H^2/2$.

In terms of a Dirac spinor $\Psi = (\omega, \pi/\sqrt{\Lambda})$ and γ-matrices, this is

$$\nabla_a\Psi + i\sqrt{\frac{\Lambda}{2}}\gamma_a\Psi = 0.$$

These equations have a 4-dimensional complex vector space $\mathbb{T}$ of solutions Z^α in any of flat, de Sitter or anti-de Sitter.

Background: symmetries from twistors

Given a twistor $Z^\alpha = (\omega^A, \pi_{A'})$ construct vectors

$$K^a = \bar{\pi}^A \pi^{A'} - \Lambda \omega^A \bar{\omega}^{A'}$$

$$L^a = \bar{\pi}^A \pi^{A'} + \Lambda \omega^A \bar{\omega}^{A'}$$

then, for $\Lambda > 0$, K^a is a space-like Killing vector and L^a is a time-like, future-pointing conformal Killing vector, which is also a gradient.

Claim: with $\Lambda \neq 0$, any conformal Killing vector (15 dim.) is the sum of a Killing vector (10 dim.) and a gradient conformal Killing vector (5 dim.) in a unique way.

Back to conserved quantities

Back to the basic identity with K^a as above:

$$\int_S T_{ab} K^a dS^b = Q[K;\Sigma] = \frac{c^2}{8\pi G} \oint_\Sigma R_{abcd} \omega^C \overline{\pi}^D \epsilon^{C'D'} d\Sigma^{ab},$$

and Q is realised as a Hermitian form on $\mathbb{T}$, conserved if T_{ab} is.

This definition carries over to curved space as *Penrose's quasi-local mass construction* – the space $\mathbb{T}$ is defined at the chosen 2-surface Σ (*2-surface twistor space*). Then, with Σ at infinity, the definition includes the previous asymptotic definitions for the cases $\Lambda \leq 0$.

BUT for $\Lambda > 0$ one loses positivity and therefore any chance of rigidity (seen e.g. by consideration of Schwarzschild-de Sitter).

Alternatively...

Still for linear theory in de Sitter space-time, take the identity as

$$Q[L;\Sigma] = \int_S T_{ab} L^a dS^b,$$

with the **conformal** Killing vector L^a. This is conserved provided T_{ab} is conserved **AND** is trace-free, and it will have positivity and rigidity.

How do we take this construction to curved space?

A modified Sen-Witten equation

Given space-like S with unit time-like normal t^a define the projected covariant derivative $D_a = \Pi_a{}^b \nabla_b$ with $\Pi_a{}^b := \delta_a{}^b - t^a t_b$. The twistor equation on S becomes

$$\tilde{D}\omega := D_{AA'}\omega_B + i\Pi_{AA'}{}^{CC'}\epsilon_{BC}\pi_{C'} = 0$$

$$\tilde{D}\pi := D_{AA'}\pi_{B'} + i\Lambda\Pi_{AA'}{}^{CC'}\epsilon_{B'C'}\omega_C = 0.$$

In general this has no solutions but its *contraction* is an elliptic system:

$$D_{AA'}\omega^A - i\Pi_{AA'}{}^{AC'}\pi_{C'} = 0$$

$$D_{AA'}\pi^{A'} - i\Lambda\Pi_{AA'}{}^{CA'}\omega_C = 0,$$

and has solutions given suitable data on $\Sigma = \partial S$.

A modified Sen-Witten identity

Introduce the Sen-Witten 2-form built from a spinor α_A as

$$\Omega_{ab}(\alpha) = \frac{i}{2}(\overline{\alpha}_{A'}\nabla_{BB'}\alpha_A - \overline{\alpha}_{B'}\nabla_{AA'}\alpha_B),$$

then (claim) provided the modified Sen-Witten equation holds one has

$$\oint_\Sigma (\Omega(\omega) + \Lambda\Omega(\overline{\pi})) = \int_S (|\tilde{D}\omega|^2 + \Lambda|\tilde{D}\pi|^2 + 4\pi G T_{ab} t^a L^b),$$

with $\Sigma = \partial S$ and $L^a = \bar{\pi}^A \pi^{A'} + \Lambda\omega^A \bar{\omega}^{A'}$ as before.

Evaluating at $\mathcal{I}^+$: 1

Suppose now that Σ lies on $\mathcal{I}^+$, then the modified Sen-Witten equation on S has a unique solution provided

- the data at Σ satisfies $\pi_{A'} = i\sqrt{2\Lambda} N_{A'A}\omega^A$, with N_a normal to $\mathcal{I}^+$,
- and ω^A satisfies the 2-surface twistor equation at Σ.

Evaluating at $\mathcal{I}^+$: 2

Conclude:

- Given Σ (and a choice of its interior) on $\mathcal{I}^+$, the construction defines a positive semi-definite form on the 2-surface twistor space $\mathbb{T}(\Sigma)$.

  • Furthermore, vanishing of the form implies that the space-time is exactly de Sitter in the domain of dependence of the interior of Σ.

- The form is naturally a real antisymmetric 4×4 matrix, M (like the space of conformal Killing vectors in de Sitter):

$$Q(Z, \overline{Z}) = M_{\alpha\beta} I^{\beta\gamma} Z^{\alpha} \overline{Z}_{\gamma},$$

  Under certain circumstances, extra structure on $\mathbb{T}(\Sigma)$ permits the definition of scalar invariants of M.

Other Approaches: 1

- Shiromizu *Phys.Rev.* **D 49** (1994) 5026. A 4-spinor approach; proves positivity of the Abbott-Deser mass on a CMC surface $K = 3\sqrt{\Lambda/3}$; later papers give examples with negative AD mass.
- Kastor and Traschen *Class.Quant.Grav.* **19** (2002) 5901 Emphasis on (approximate) conformal isometries and modification of Sen-Witten PMT; they define a single quantity as though for the CKV $X = R(t)\frac{\partial}{\partial t}$ in FRW.

Other Approaches: 2

- Ashtekar, Bonga and Kesavan, *arXiv 1409.3816, 1506.06152, 1510.05593*
  Largely concerned with linear theory about de Sitter in a 'Poincaré patch':
  $$g = dt^2 - e^{2Ht}(dx^2 + dy^2 + dz^2);$$
  Paper III proves a quadropole formula.
- Chruściel and Ifsits *arXiv 1603.07018*
  An effective definition tailored to characteristic hypersurfaces in dimension $n + 1$.