Problem sheet 4, Maths B7.2a (Physics B4) MT 2006

1. Let P and K be constant 4-vectors and let X denote the displacement 4-vector from some fixed origin. Put $\Omega = g(K, X)$. Show that

$$\Phi = P \cos(\Omega)$$

is a 4-potential for an electromagnetic field in the Lorenz gauge if and only if K is null and orthogonal to P. Find and interpret the corresponding electric and magnetic fields.

2. Deduce from

$$\frac{\mathrm{d}}{\mathrm{d}t}(m\gamma \boldsymbol{u}) = e(\boldsymbol{E} + \boldsymbol{u} \wedge \boldsymbol{B})$$

that

$$\frac{\mathrm{d}}{\mathrm{d}t}(m\gamma c^2) = e\boldsymbol{E}.\boldsymbol{u}\,.$$

[Remember that $\gamma u = c\sqrt{\gamma^2 - 1}$.]

3. Show that the *dual electromagnetic field*

$$F^* = \begin{pmatrix} 0 & cB_1 & cB_2 & cB_3 \\ -cB_1 & 0 & -E_3 & E_2 \\ -cB_2 & E_3 & 0 & -E_1 \\ -cB_3 & -E_2 & E_1 & 0 \end{pmatrix}$$

is a tensor. [Hint: deal separately with (i) the standard Lorentz transformation and (ii) rotations of the x, y, z axes. Alternatively use the results of the next exercise by finding the relationship between F^{*ab} and $\varepsilon_{abcd}F^{cd}$.]

By considering $F_{ab}F^{*ab}$, show that $\boldsymbol{B}.\boldsymbol{E}$ is an invariant.

4. The *alternating symbol* is defined by

$$\varepsilon_{abcd} = \begin{cases} 1 & \text{if } abcd \text{ is an even permutation of } 0123 \\ -1 & \text{if } abcd \text{ is an odd permutation of } 0123 \\ 0 & \text{otherwise.} \end{cases}$$

Show that if T, X, Y, Z are 4-vectors, then

$$\varepsilon_{abcd}T^aX^bY^cZ^d = \begin{vmatrix} T^0 & X^0 & Y^0 & Z^0 \\ T^1 & X^1 & Y^1 & Z^1 \\ T^2 & X^2 & Y^2 & Z^2 \\ T^3 & X^3 & Y^3 & Z^3 \end{vmatrix}$$

Deduce that $\varepsilon_{abcd}T^aX^bY^cZ^d$ is invariant under Lorentz transformations and hence that ε_{abcd} is a tensor of type $(0, 4)$ with the same components in every inertial coordinate system.

Write down the values of components of the contravariant tensor ε^{abcd}.

Show that $\varepsilon_{abcd}\varepsilon^{abcd} = -24$ and that $\varepsilon_{abcd}\varepsilon^{abce} = -6\delta^e_d$.

5. A particle of rest mass m, charge e and initial velocity $\boldsymbol{v}$ relative to some ICS is moving in a constant magnetic field $\boldsymbol{B}$ perpendicular to $\boldsymbol{v}$. Show that the particle moves in a circle and that the proper time on the particle worldline that elapses on each circuit is $2\pi m/eB$.

   Deduce that if a particle of rest mass m and charge e moves through a constant uniform electromagnetic field in which $\boldsymbol{B}.\boldsymbol{E} = 0$ and $\boldsymbol{E}.\boldsymbol{E} < c^2\boldsymbol{B}.\boldsymbol{B}$, then after the elapse of proper time $2\pi mc/e\sqrt{c^2\boldsymbol{B}.\boldsymbol{B} - \boldsymbol{E}.\boldsymbol{E}}$ measured along the worldline of the particle, its velocity relative to the ICS is equal to its initial velocity.

6. Show that Maxwell's equations without sources can be derived from the Lagrangian density $L = -\frac{1}{4}\epsilon_0 F_{ab}F^{ab}$, treating the four-potential Φ^a as the field variable.

   Let J^a be a given current 4-vector. Show that Maxwell's equations with this source can be derived from a variational principle by adding a constant mutiple of $\Phi_a J^a$ to the Lagrangian density. Show that provided that f behaves appropriately at infinity, then the action (the integral of the Lagrangian density over space-time) is invariant under gauge transformations $\Phi_a \mapsto \Phi_a + \nabla_a f$.