Twistors and conformal field theory

I shall begin with a brief comparison of twistors in two and four dimensions. In the following table, \( A \) and \( A' \) are usual 2-spinor indices and we write \( \mathcal{P}^A = \mathcal{P}(C^A), \mathcal{P}^{A'} = \mathcal{P}(T^{A'}) \), etc.

<table>
<thead>
<tr>
<th></th>
<th>2 dimensions</th>
<th>4 dimensions</th>
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</thead>
<tbody>
<tr>
<td>real space-time</td>
<td>( M_2^* = S^1 \times S^1 )</td>
<td>( M^* = S^3 \times S^1 )</td>
</tr>
<tr>
<td>C space-time</td>
<td>( C^A, C^A, C_A, C_A' )</td>
<td>( \mathbb{C}P^3 )</td>
</tr>
<tr>
<td>twistor spaces</td>
<td>( \mathcal{E}<em>{AB}, \mathcal{E}</em>{AB}', \text{etc.} )</td>
<td>( T^+, T^-, T^0, T^i )</td>
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<tr>
<td>“( \varepsilon )-object”</td>
<td>( \lambda^A \to \tilde{\lambda}^A, (\tilde{\lambda}^0, \tilde{\lambda}^i) = (\lambda^0, \lambda^i) )</td>
<td>( \varepsilon_{\alpha \alpha}', \text{etc.} )</td>
</tr>
<tr>
<td>reality structure</td>
<td>( \mathcal{C}^A_2 = \mathcal{P}^A \times \mathcal{P}^{A'} )</td>
<td>( \mathcal{C}^A_4 = \mathcal{P}^A \times \mathcal{P}^{A'} )</td>
</tr>
<tr>
<td>twistor correspondences</td>
<td>( \mathcal{C}^A_2 = \mathcal{P}^A \times \mathcal{P}^{A'} \to \mathcal{P}^A )</td>
<td>(well-known)</td>
</tr>
</tbody>
</table>

Remark

1. The conformal structure of \( \mathcal{C}^A_2 \) is supposed given by its product structure, the two families of rulings being the two families of null geodesics.
2. The data in the above table are consistent with the standard way of thinking of twistors as spinors for the conformal group \([1]\). Case is required in dimension 2, however, since the group of conformal motions of \( M_2^* \) is infinite dimensional. The “correct” conformal group, \( O(2,2) \) is characterized as being the group of holomorphic
conformal motions of $C\mathbb{H}_2$ which carry the real slice $M^*_2$ to itself.

3. Note the differences in the $\mathcal{E}$-objects and the reality structures in the two cases: and note also that although the twistor for two dimensions look like ordinary 2-component spinors, the conjugation is different (because we're interested in $O(2,2)$ instead of $O(1,3)$). Of course one usually uses $\epsilon_{\alpha\beta}$ to eliminate all primed twistor indices: for example, the familiar conjugation $Z^\alpha \rightarrow \overline{Z}_\alpha$ is given by $\overline{Z}_\alpha = \epsilon^{\alpha\beta} \epsilon_{\alpha\beta}$.

The basic ingredient of a conformally invariant quantum field theory (CFT) is a Hilbert space $H$ of states. For conformal invariance one tends to take $H$ to be a space of positive-energy solutions $W^+$ to some massless field equations or a Fock space modelled on $W^+$. In both dimensions we are considering the twistor construction of $W^+$ in particular elegant:

$$\Gamma(P^+, O(\mathbb{C}^{\cdot-1}))$$

Here I have written $P^+$ for the closure of $\mathcal{P}^+$ and $P_0^+$ for $\mathcal{P}_N$. Similarly, in the two-dimensional picture I'm thinking of $P^+$ and $P^-$ as the closed hemispheres. Strictly speaking, $W^+$ for the two-dimensional case should be the holomorphic sections over $P^+$ modulo the global sections.

Thus in each case, the top half of twistor
space defines $W^+$ in a natural way. An alternative way of constructing $W^+$ is to consider elementary states which give an orthogonal basis for $W^+$ (see [2] for more details). In each case the elementary states are defined on the punctured twistor space: when I speak of a punctured Riemann surface I shall always mean that finite number of points have been removed, but by "punctured complex 3-manifold" I shall mean that a finite number of (non-intersecting) projective lines have been removed. To get a basis for $W^+$ the only condition is that the puncture must be in the interior of the bottom half of twistor space.

\[ T^P \{ P \} \quad 0 \}

\[ H^1(\mathbb{P}^2, O(k-2)) \quad 0\}

All this is fine for a trivial free CFT: but what about interactions? In two dimensions there are two (nearly equivalent) ways to proceed [3]. In the present language one replaces the Riemann surface

involved in the "free theory" by an arbitrary compact Riemann surface $X$ with (oriented, parametrized) boundary circles or with punctures (with holomorphic coordinates near
each puncture). For such surfaces, there are natural ways of gluing them together (for these one needs parametrized boundaries or coordinates near the punctures). For punctures P and Q with z a holomorphic coordinate vanishing at P and w a holomorphic coordinate vanishing at Q, the gluing is defined by the identification zw = 1. A reinterpretation in terms of spinor (i.e. two-dimensional twistor) contour integrals can be found in A.P.H.'s article in this TN.

On the other hand, R.P. (this TN) shows how the analogous constructions can be made in four dimensions by gluing bits of twistor space together across PN boundaries or punctures. It is an interesting feature of the higher-dimensional case that there is less freedom in gluing boundaries together (both assumed to be copies of PN) on account of the rigid CR structure of PN. This is in contrast to the infinite-dimensional Diff(S¹) freedom which appears in two dimensions. Thus by gluing pieces of twistor spaces together we can construct higher-dimensional analogues of Riemann surfaces together and glue them together.

Although this extension of the analogy is rather satisfactory it is only part of what is required for the construction of an interacting CFT.

Returning to the two-dimensional case, one selects a Hilbert space H and imagines attaching a copy of H to each boundary circle or puncture of the Riemann
surface $X$. (Actually, for what I'm about to say, one attaches $H = H^*$ to any negatively oriented boundary circle.) Then to $X$ one assigns an amplitude on the tensor product of the attached Hilbert spaces. This assignment is to satisfy various conditions, the most important being its "naturality" under gluing operations.

How such an assignment of amplitudes could be achieved in the 4-dimensional case is not known at present.

If the "punctures" approach were adopted, one way to proceed would be to assign an amplitude to $\mathcal{P}^*$ (punctures) and to give a rule for the effect of gluings on the amplitudes. In that way, one could assign an amplitude to all of $\mathcal{P}^*$'s glued-together twistor space. One would expect vertex operators (see T.S.T. in this TN) to play an important part in such an approach.

In the two dimensional case, a popular choice for $H$ is a Fock space on some $\mathcal{M}$. Then in the boundary picture the amplitudes can nearly be constructed from the subspace structure

$$\mathcal{O}(X) \subset C^\infty(\mathcal{M})$$

induced by restriction of holomorphic functions on $X$ to the boundary. (See G.B.S. - in person - for more details.) The "physical interpretation" of such a theory is then in terms of scattering of strings, the different Fock-space sectors being reinterpreted (roughly speaking) as giving the different modes of the string. It seems quite likely that such a Fock-space theory could be built in four dimensions too, but one would then want a non-stringy reinterpretation of the Fock-space sectors. Is it possible that these could represent different particles, along the lines of the twistor-particle programme? If on the other hand one could construct...
a two-dimensional theory with \( H = W^+ \) that might give some clues for a four-dimensional theory with \( H = W^+ \).

Any possible link with twistor diagram theory is obscure to the author (but see A.P.H. in this TN). Notable by its absence from the above is dual twistor space. It may be, however, that the proposed extension to four-dimensions should be modified to allow gluings of twistor space to dual twistor space, even though the two-dimensional theory is formulated solely in terms of one twistor space. One reason for believing such a thing is the differences in the actions of the reality structures alluded to under \((3)\) above. At that level, the two 2-dimensional twistor spaces are, after all, completely unrelated to each other, whereas the twistor spaces are linked by the conjugation in four dimensions.

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References


For “punches” approach: C. Vafa: Conformal theories and punctured surfaces, preprint.