Twistors and conformal field theory

I shall begin with a brief comparison of twistors in two and four dimensions. In the following table, A and A' are usual 2-spinor indices and we write $\mathbb{P}^A = \mathbb{P}(\mathbb{C}^A)$, $\mathbb{P}^\alpha = \mathbb{P}(\mathbb{T}^\alpha)$, etc.

	2 dimensions	4 dimensions
real space-time	$M_2^\# = S^1 \times S^1$	$M^\# = S^3 \times S^1$
$\mathbb{C}$ space-time	$\mathbb{C}M_2^\# = \mathbb{P}^A \times \mathbb{P}^{A'}$	$\mathbb{C}M^\# =$ quadric in $\mathbb{CP}^5$
twistor spaces	$\mathbb{C}^A, \mathbb{C}^{A'}, \mathbb{C}_A, \mathbb{C}_{A'}$	$\mathbb{T}^\alpha, \mathbb{T}^{\alpha'}, \mathbb{T}_\alpha, \mathbb{T}_{\alpha'}$
"ε-object"	$\varepsilon_{AB}, \varepsilon_{A'B'}$, etc	$\varepsilon_{\alpha\alpha'}$, etc
reality structure	$\lambda^A \mapsto \hat{\lambda}^A, (\hat{\lambda}^0, \hat{\lambda}^1) = (\bar{\lambda}^1, -\bar{\lambda}^0)$	$Z^\alpha \mapsto \bar{Z}^{\alpha'}$
twistor correspondences	$\mathbb{C}M_2^\# = \mathbb{P}^A \times \mathbb{P}^{A'} \to \mathbb{P}^A$ $\mathbb{C}M_2^\# = \mathbb{P}^A \times \mathbb{P}^{A'} \to \mathbb{P}^{A'}$	(well-known)

Remarks

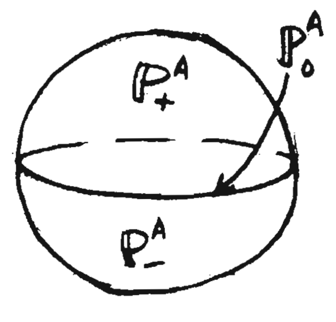
1. The conformal structure of $\mathbb{C}M_2^\#$ is supposed given by its product structure, the two families of rulings being the two families of null geodesics.
2. The data in the above table are consistent with the standard way of thinking of twistors as spinors for the conformal group [1]. Care is required in dimension 2, however, since the group of conformal motions of $M_2^\#$ is infinite dimensional. The "correct" conformal group, $O(2,2)$ is characterized as being the group of holomorphic

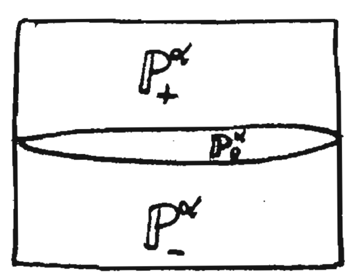
conformal motions of $\mathbb{C}M_2^{\#}$ which carry the real slice $M_2^{\#}$ to itself.

3. Note the differences in the ε-objects and the reality structures [1] between the two cases: and note also that although the twistors for two dimensions look like ordinary 2-component spinors, the conjugation is different (because we're interested in $O(2,2)$ instead of $O(1,3)$). Of course one usually uses $\varepsilon_{\alpha\alpha'}$ to eliminate all primed twistor indices: for example the familiar conjugation $Z^\alpha \mapsto \bar{Z}_\alpha$ is given by $\bar{Z}_\alpha = \bar{Z}^{\alpha'}\varepsilon_{\alpha\alpha'}$.

The basic ingredient of a conformally invariant quantum field theory (CFT) is a Hilbert space H of states. For conformal invariance one tends to take H to be a space of positive-energy solutions W^+ to some massless field equations or a Fock space modelled on W^+. In both dimensions we are considering the twistor construction of W^+ is particularly elegant:

decomposition of projective twistor-space under reality structure





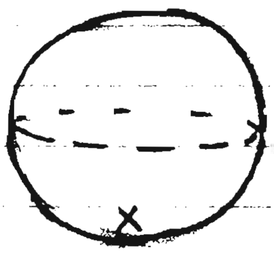
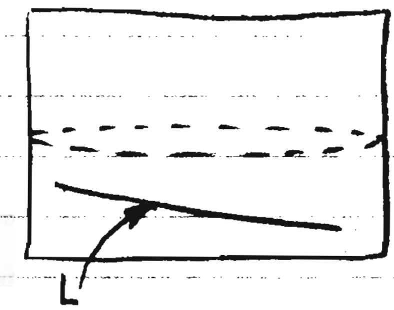
Typical example of W^+

$\Gamma(\mathbb{P}_+^A, \mathcal{O}(k-1))$

$H^1\left(\mathbb{P}_+^\alpha, \begin{array}{c}\mathcal{O}(k-2)\\ \oplus \\ \mathcal{O}(-k-2)\end{array}\right)$

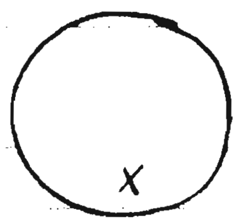
Here I have written $\mathbb{P}_+^\alpha$ for the <u>closure</u> of $\mathbb{PT}^+$ and $\mathbb{P}_0^\alpha$ for $\mathbb{PN}$. Similarly, in the two-dimensional picture I'm thinking of $\mathbb{P}_+^A$ and $\mathbb{P}_-^A$ as the <u>closed</u> hemispheres. Strictly speaking, W^+ for the two-dimensional case should be the holomorphic sections over $\mathbb{P}_+^A$ modulo the global sections.

Thus in each case, the top half of twistor space defines W^+ in a natural way. An alternative way of constructing W^+ is to consider <u>elementary states</u> which give an orthogonal basis for W^+ (see [2] for more details). In each case the elementary states are defined on the <u>punctured</u> twistor space: when I speak of a punctured Riemann surface I shall always mean that a finite number of points have been removed, but by "punctured complex 3-manifold" I shall mean that a finite number of (non-intersecting) projective lines have been removed. To get a basis for W^+ the only condition is that the puncture must be in the interior of the bottom half of twistor space.

punctured twistor space		
space of elementary states	$\Gamma(\mathbb{P}^1 - \{pt\}, \mathcal{O}(k-1))$	$H^1\left(\mathbb{P}^3 - L, \begin{matrix}\mathcal{O}(k-2)\\ \oplus \\ \mathcal{O}(-k-2)\end{matrix}\right)$

All this is fine for a twistorial free CFT: but what about interactions? In two dimensions there are two (nearly equivalent) ways to proceed [3]. In the present language one replaces the Riemann surface

or

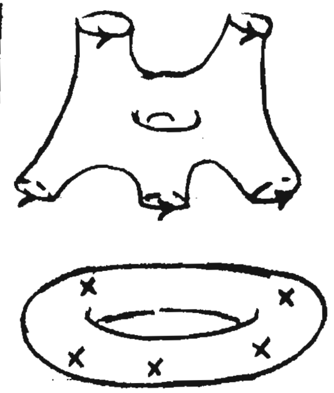


involved in the "free theory" by an arbitrary compact Riemann surface X with (oriented, parametrized) boundary circles or with punctures (with holomorphic coordinates near each puncture). For such surfaces, there are natural ways of glueing them together (for these one needs parametrized boundaries or coordinates near the punctures). For punctures P and Q with z a holomorphic coordinate vanishing at P and w a holomorphic coordinate vanishing at Q, the glueing is defined by the identification $zw = 1$. A reinterpretation in terms of spinor (i.e. two-dimensional twistor) contour integrals can be found in A.P.H.'s article in this TN.

On the other hand, R.P. (this TN) shows how the analogous constructions can be made in four dimensions by glueing bits of twistor space together across $\mathbb{P}N$ boundaries or punctures. It is an interesting feature of the higher-dimensional case that there is <u>less</u> freedom in glueing boundaries together (both assumed to be copies of $\mathbb{P}N$) on account of the rigid CR structure of $\mathbb{P}N$. This is in contrast to the infinite-dimensional $\mathrm{Diff}(S^1)$ freedom which appears in two dimensions. Thus by glueing pieces of twistor spaces together we can construct higher dimensional analogues of Riemann surfaces together and glue them together.

"boundary" approach

"puncture" approach



Complex compact 3-manifolds with $\mathbb{P}N$s as boundary.

Complex compact 3-manifolds with (line) punctures.

Although this extension of the analogy is rather satisfactory it is only part of what is required for the construction of an interacting CFT.

Returning to the two-dimensional case, one selects a Hilbert space H and imagines attaching a copy of H to each boundary circle or puncture of the Riemann surface X. (Actually, for what I'm about to say, one attaches $\bar{H} = H^*$ to any negatively oriented boundary circle.) Then to X one assigns an <u>amplitude</u> on the tensor product of the attached Hilbert spaces. This assignment is to satisfy various conditions, the most important being its "naturality" under glueing operations.

How such an assignment of amplitudes could be achieved in the 4-dimensional case is not known at present.

If the "punctures" approach were adopted, one way to proceed would be to assign an amplitude to $\mathbb{P}^\alpha - \{3 \text{ lines}\}$ and to give a rule for the effect of glueings on the amplitudes. In that way, one could assign an amplitude to all of R.P.'s glued-together twistor spaces. One would expect vertex operators (see T.S.T. in this TN) to play an important part in such an approach.

In the two dimensional case, a popular choice for H is a Fock space on some W.[+] Then in the "boundary" picture the amplitudes can nearly be constructed from the subspace structure

$$\mathcal{O}(X) \subset C^\infty(\partial X)$$

induced by restriction of holomorphic functions on X to the boundary. (See G.B.S. – in person – for more details.) The "physical interpretation" of such a theory is then in terms of scattering of strings, the different Fock-space sectors being reinterpreted (roughly speaking) as giving the different modes of the string. It seems quite likely that such a Fock-space theory could be built in four dimensions too, but one would then want a non-stringy reinterpretation of the Fock-space sectors. Is it possible that these could represent <u>different</u> particles, along the lines of the twistor particle programme?

If on the other hand one could construct a two-dimensional theory with $H = W^+$, that might give some clues for a four-dimensional theory with $H = W^+$.

Any possible link with twistor diagram theory is obscure to the author (but see A.P.H. in this TN). Notable by its absence from the above is dual twistor space. It may be, however, that the proposed extension to four dimensions should be modified to allow glueings of twistor space to dual twistor space, even though the two-dimensional theory is formulated solely in terms of one twistor space. One reason for believing such a thing is the differences in the actions of the reality structures alluded to under (3) above. At that level, the two 2-dimensional twistor spaces are, after all, completely unrelated to eachother, whereas the twistor spaces are linked by the conjugation in four dimensions.

I acknowledge many useful talks with R.J.B., E.D., A.P.H., R.P., G.B.S., T.S.T.

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References

[1] R.P. and W.R. Spinors and space-time vol. 2, Appendix.
[2] M.G.E. and A.M.P. On the density of elementary states, TN 16.
[3] For "boundary" approach: G.B. Segal: The definition of conformal field theory, to appear.
For "punctures" approach: C. Vafa: Conformal theories and punctured surfaces, preprint.