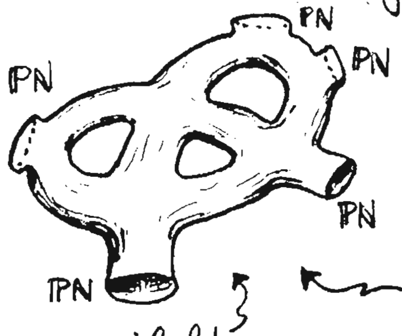
Pretzel Twistor Spaces

In an accompanying article in this $\mathbb{T}\mathbb{N}$, M.A.S. has provided a very appealing suggestion for combining some of the attractive features of string theory / conformal field theory with twistor theory. As an analogue of the (bounded) Riemann surfaces of str./conf. fld. theory (pretzels), one imagines a complex



manifold — say of 3 dimensions or 4 dimensions, according to whether we are considering projective or non-projective twistors — with a boundary consisting of a number of copies of $\mathbb{PN}$ or $\mathbb{N}$, as the case may be. (See also T.S.T.'s article for a suggestion whereby these $\mathbb{PN}$s shrink down to lines.) The most appropriate formulation of this idea remains somewhat unclear as of now, but we might think in terms of some sort of analogue of the Fock spaces that in str./conf. fld. th. are to be specified at each disconnected portion of the boundary — say, at each "end". Then amplitudes could be obtained when elements of the appropriate Fock spaces are specified at each end.

From the point of view of twistor theory, a Fock space (in its normal physical interpretation) might seem inappropriate. For reasons of wishing to tie up this idea with twistor diagram theory — ideas due to A.P.H. (see this $\mathbb{T}\mathbb{N}$) — and of exploiting the relevant crossing symmetry, spin-statistics, duality, etc., we might prefer to think of the "ends" as corresponding to individual particles, rather than states involving an indefinite number of particles as described by Fock space elements.

The twistorial description of Fock space elements would be by certain elements of

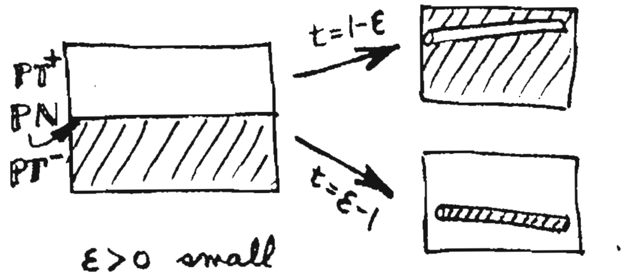
$$\mathbb{C} \oplus H^1(\mathbb{T}^+, \mathcal{O}) \oplus H^2(\mathbb{T}^+ \times \mathbb{T}^+, \mathcal{O}) \oplus H^3(\mathbb{T}^+ \times \mathbb{T}^+ \times \mathbb{T}^+, \mathcal{O}) \oplus \dots$$

i.e. by collections of "twistor functions"

$$f_0\,,\; f_1(Z^\alpha)\,,\; f_2(Z^\alpha, Y^\alpha)\,,\; f_3(Z^\alpha, Y^\alpha, X^\alpha)\,,\; \dots$$

which are, respectively "0-functions", 1-functions, 2-functions, 3-functions, etc., where the n-function f_n is the twistor wave function for a state consisting of n massless particles. (The "0-function" f_0 would simply be an element of $\mathbb{C}$, defining the amplitude for zero particles.) Instead of this, we might well prefer to think of $f_0, f_1, f_2, f_3, \ldots$ as referring to the functions of several twistors which come up in the twistor particle programme. Thus, to a "first approximation", f_1 might describe the amplitude for the state of a massless particle, f_2 the state of a lepton (?), f_3 the state of an "ordinary" hadron (??), f_4, etc., for more "exotic" hadrons (???) (and f_0 for a "nothing" ..?...?), but an actual massive particle might tend to have small contributions all along the line. We expect f_1 to be a 1-function and suspect that f_2 might perhaps be a relative 1-function (or 2-function?) relative to some sort of diagonal locus in $\mathbb{T} \otimes \mathbb{T}$, conformal invariance being broken at this stage (so mass and $I_{\alpha\beta}$ come in). Perhaps all the f_i are relative 1-functions (or 2-functions) or something (whence $f_0 \equiv 0$). Perhaps, as with strings, the higher fs describe higher "modes" — oscillation of strings being replaced by deformations of $\mathbb{P}N$?? Perhaps, in accordance with an idea floated by G.B.S., they refer to different jet bundles over $(\mathbb{P})N$??

All this is extremely vague and speculative, as yet. Let us backtrack a little and try to see whether there are, indeed, any interesting "pretzel" twistor spaces. As with str./conf. theories, closed pretzel spaces are of particular interest. the simplest way to construct such a twistor space (considering the case of pretzelized $\mathbb{P}\mathbb{T}$s only) would seem to be the following. Consider first the two limiting projective motions of $\mathbb{P}\mathbb{T}$:

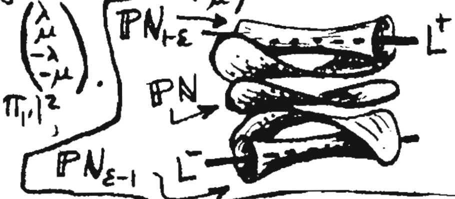


These may be achieved by

$$Z^\alpha \nearrow \begin{bmatrix}\omega^0\\ \omega^1\\ \pi_{0'}\\ \pi_{1'}\end{bmatrix} \longmapsto \begin{bmatrix}1&0&t&0\\0&1&0&t\\t&0&1&0\\0&t&0&1\end{bmatrix}\begin{bmatrix}\omega^0\\ \omega^1\\ \pi_{0'}\\ \pi_{1'}\end{bmatrix} = \begin{bmatrix}\omega^0+t\pi_{0'}\\ \omega^1+t\pi_{1'}\\ \pi_{0'}+t\omega^0\\ \pi_{1'}+t\omega^1\end{bmatrix} \swarrow Z^\alpha_t$$

with $t \in (-1,1)$, the two limiting cases being $t \to 1$, $t \to -1$, respectively.

The 5-surface into which $\mathbb{P}N$ is carried (i.e. given by $\underset{-t}{Z}^{\alpha}\underset{-t}{\bar{Z}}_{\alpha}=0$) is another copy of $\mathbb{P}N$ (identical with it as a CR-manifold) — call it $\mathbb{P}N_t$. As $t\to 1$, this 5-surf. closes in on the line $L^+\subset \mathbb{P}\mathbb{T}^+$ given by $Z^\alpha = \begin{pmatrix}\lambda\\ \mu\\ \lambda\\ \mu\end{pmatrix}$, and as $t\to -1$ it closes in on $L^-\subset\mathbb{P}\mathbb{T}^-$, given by $\begin{pmatrix}\lambda\\ \mu\\ -\lambda\\ -\mu\end{pmatrix}$.



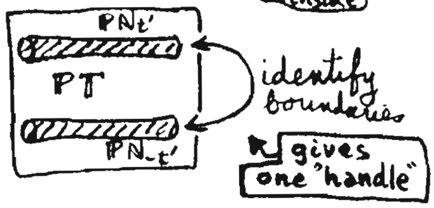
Note that if we define $|Z|^2=|\omega^0|^2+|\omega^1|^2+|\pi_{0'}|^2+|\pi_{1'}|^2$, then $|\underset{t}{Z}|^2=(1+t^2)|Z|^2+2tZ^\alpha\bar{Z}_\alpha$ and

$\underset{t}{Z}^{\alpha}\underset{t}{\bar{Z}}_{\alpha}=(1+t^2)Z^\alpha\bar{Z}_\alpha+2t|Z|^2$. Hence $\mathbb{P}N_t$ is defined by

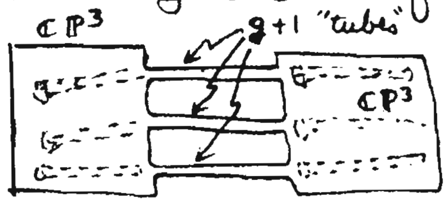
$$Z^\alpha\bar{Z}_\alpha : |Z|^2 = 2t : 1+t^2 .$$

$\mathbb{P}N_{1-\varepsilon}$ (resp. $\mathbb{P}N_{\varepsilon-1}$) is the boundary of a tubular neighbourhood of L^+ (resp. L^-).

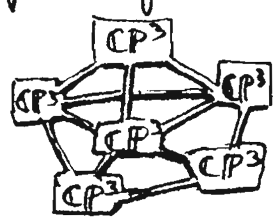
The simplest pretzel space $\mathcal{P}_1$ is given by identifying Z^α with $\underset{t}{Z}^{\alpha}$ for some fixed t in $(0,1)$ (and hence, up to proportionality, Z^α is also identified with $\underset{\tanh(n\tau)}{Z}^{\alpha}$ where $t=\tanh\tau$, for all $n\in\mathbb{Z}$). This can also be achieved by drilling out the regions above (inside) $\mathbb{P}N_{t'}$ and below (inside) $\mathbb{P}N_{-t'}$ (where $t'=\tanh\frac{1}{2}\tau$) and identifying the boundaries. There is actually some freedom in how such an identification is made, and this allows different pretzel twistor spaces to be built in this way.



The spaces $\mathcal{P}_1$ are the anologues of <u>toruses</u>, i.e. of Riemann surfaces of genus 1. For higher genus (genus g), we can carry out several such identifications simultaneously to give a "$\mathbb{C}\mathbb{P}^3$ with g handles". Perhaps easier to visualize, but equivalent, is to take <u>two</u> copies of $\mathbb{C}\mathbb{P}^3$ and to identify across from one to the other.



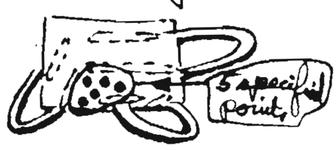
We could also do, etc.,



but this gives us no more generality. (To see this, clip all the tubes necessary to make a tree; then fit all the $\mathbb{C}\mathbb{P}^3$s inside <u>one</u> of them and we are back with the case of "handles" as before, when we reglue where we had clipped.)

The analogies between these spaces and <u>Riemann</u>

surfaces are quite striking. In particular, we can ask for the dimension m of the moduli space for pretzel twistor spaces of genus g (i.e., how many complex parameters are needed to characterize a $\mathcal{P}_g$, where a $\mathcal{P}_g$ is a $\mathbb{CP}^3$ with g "handles" of this type). (I don't know whether this "genus" corresponds to something standard in algebraic geometry.) Recall that for Riemann surfaces the answer (due to Riemann) is $3g-3$, except when $g=0$ or 1 — where for $g=0$ the answer is 0 and for $g=1$ it is 1. For $\mathcal{P}_g$ the answer turns out to be precisely 5 times as large — with the single exception that for $\mathcal{P}_1$, $m=3$ (instead of 5). The proof is similar to that for Riemann surfaces. Think first of a labelled $\mathbb{CP}^3$. The labelling can be achieved by specifying 5 points in general position on $\mathbb{CP}^3$. There are 15 complex degrees of freedom in the specification of each pipe.



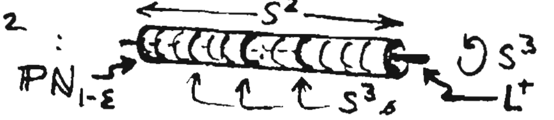
(Or do it with

, which may be a little easier to visualize.) The 15 comes from the size of the projective group on $\mathbb{CP}^3$ $(15 = 4^2-1)$. We have $15g$ for the number of parameters needed to define a labelled $\mathcal{P}_g$. We have to factor out by the freedom in doing the labelling — which is 15 parameter's worth unless, in the generic case, there are continuous (holomorphic) symmetries of $\mathcal{P}_g$. We do not factor out by the motions of the labelling corresponding to a symmetry. Suppose that there are d dimensions of symmetries. Then we get

$$m = 15g - (15-d).$$

When $g=0$ then clearly $d=15$, so $m=0$. When $g=1$ then a direct argument shows that $m=3$ (whence $d=3$ in the generic case). (This argument is: $\mathcal{P}_1$ is defined by $Z^\alpha \equiv T^\alpha{}_\beta Z^\beta$, up to proportionality, for some fixed $T^\alpha{}_\beta$. The ratios of the eigenvalues of $T^\alpha{}_\beta$ give the moduli of $\mathcal{P}_1$.) It is not hard to see that $d=0$ whenever $g \geq 1$ (as with Riemann surfaces), so $m = 15g-15$ in these cases.

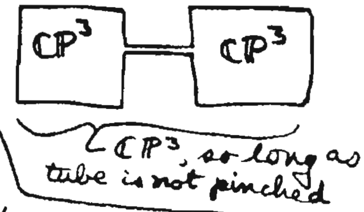
I should remark that there is a subtlety involved in the gluing of the pipes together ("handles"). When we think of $\mathbb{P}N_{1-\varepsilon}$ surrounding L^+, we think of it as an $S^2 \times S^3$ for which the S^3s surround the various points of L^+ (in directions normal to L^+) and where the different points of L^+ give the S^2:

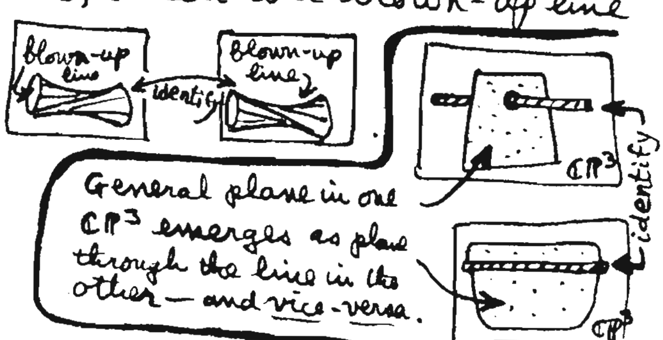


. If we glue this to $\mathbb{P}N_{\varepsilon-1}$ we must do so in such a way that the small S^3s of $\mathbb{P}N_{1-\varepsilon}$ are stretched the length of $\mathbb{P}N_{\varepsilon-1}$,

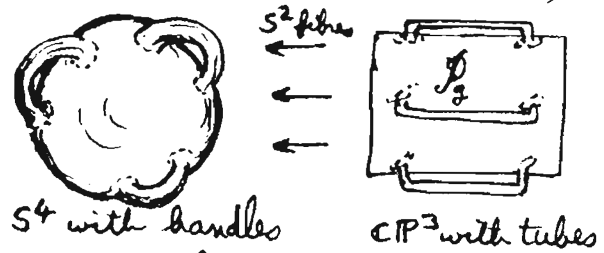
and vice versa. This is possible because of a topological relation which I write symbolically as $S^2\times S^3 \cong S^3\times S^2$, each side being a circle bundle over $S^2\times S^2$, but where the twist can be transferred from being over the second S^2 factor to over the first. (This is the Clifford-Hopf bundle of S^1 over S^2 to give S^3.) If we take planes through L^-, they sweep out one family of S^3s on each $\mathbb{P}N_t$ ("small S^3s" when $t=1-\varepsilon$, and running the "length" of $\mathbb{P}N_t$ when $t=\varepsilon-1$); planes through L^+ sweep out the other family of S^3s ("long" when $t=1-\varepsilon$ and "small" when $t=\varepsilon-1$). The relation "$S^2\times S^3 \cong S^3\times S^2$" can also be seen by examining pairs of orthogonal unit vectors at the origin of $\mathbb{R}^4$. Fix attention on one vector: it sweeps out an S^3 while the other gives a (trivial) bundle over it (trivial since S^3 is parallelizable). Then think of the vectors in the other order.

If we consider a thin tube, where the tube narrows down to nothing, we get in the limit the space (considered by twistorians in connection with null lines on $\mathbb{C}\mathscr{I}$, and by S.K.D. in a context similar to the present one) which is two $\mathbb{CP}^3$s joined along a quadric, which is a blown-up line in each, and identified with generator systems reversed:





The topology of $\mathscr{P}_g$ is an S^2 bundle over S^4 with g handles ($S^3\times\mathbb{R}$ handles).



← These spaces $\mathscr{P}_g$ are ones where there are $\mathbb{CP}^1$s with neighbourhoods which are identical with portions of $\mathbb{CP}^3$ — and are twistor spaces of conformally flat 4-spaces. I think they are the only ones.

Much more can be said. Of course, in line with non-linear graviton constructions we shall want to deform these spaces; also to deform the CR-structure of $\mathbb{P}N$, etc. Work in progress.

Thanks to M.A.S., A.P.H., G.B.S., T.S.T., E.D., R.B., S.K.D.

~ Roger Penrose