A possible role of vertex operators in Singer's picture of 4-dim CFT

Singer proposes (see this issue of TN) that one can view the Riemann surfaces of 2-dim. conformal field theory as a 2-dim. analogue of twistor space. This leads to the further proposal that 4-dim. CFT can be obtained through replacing the Riemann surface (with boundary a collection of S^1) by a complex 3-dim. manifold (with boundary a collection of $\mathbb{P}N$).

Now one way to make contact with the physical world of interactions in the 2-dim. theory is to introduce vertex operators. For simplicity, consider a cylinder with two bounding circles. If we think of these two circles as representing an incoming and an outgoing state (just as in string theory), then by conformal invariance the bounding circles can be shrunk to points (not entirely clear how, but universally accepted by physicists) and the Riemann surface becomes a sphere with two punctures:

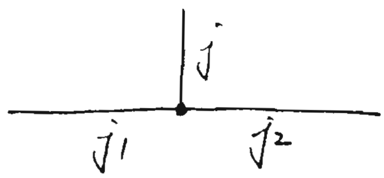
The "physics" is then represented by local operators called vertex operators inserted at these points. These vertex operators keep track of the momenta, positions and quantum numbers of the particles states. For spinless particles one can take e.g. $V_0(k,z) = e^{ik\cdot X}$, and for spin 2, $V_2(k,z) = \partial_a X^b \partial^a X^c \, e^{ik\cdot X}$, where k = momentum and $X = X(z)$ = coordinate. To get the N-point amplitude one then takes the vaccuum expectation value of the product of N such vertex operators:

$$A = \left\langle \prod_{i=1}^{N} V_{J_i}(k_i, z_i) \right\rangle .$$

E.g. N=4 leads to the well-known Veneziano amplitude involving hypergeometric functions. Depending on the particular problem, these functions of k and z can be integrated w.r.t. either variable, and the resultant functions are also called vertex operators.

The twistor picture is tantalizingly similar. If we take a line in $\mathbb{PT}$ we can choose a tubular neighbourhood of it to make it look like a standard $\mathbb{PN}$. (See RP's article in this issue). Since these $\mathbb{PN}$ are to play the role of the S^1 in the 2-dim. theory, the lines whose neighbourhoods they are then correspond to the points (or vertices) at which one can insert vertex operators. But a line in $\mathbb{PT}$ corresponds to a point in 4-dim. "space-time", and since such a line is in a $\mathbb{PN}$ the point is "real". So this looks the right object to which one can attach vertex operators. The obvious suggestion is to consider elementary states. Unfortunately I do not know how to do this concretely at present. Perhaps one can think of a different kind of "Penrose transform".

A different way to look at vertex operators is in the representation theory of Diff S^1. They correlate different Verma modules rather like the way Clebsch-Gordon coefficients connect different spins. In fact, I think that it is "pictorially" correct to say that vertex operators are souped-up continuum versions of Clebsch-Gordon coefficients which are just numbers. Roughly, let ℓ be a non-negative integer and j a half-integer s.t. $0 \leq 2j \leq \ell$. Consider the affine Lie algebra of $sl(2,\mathbb{C})$, and denote by V_j the subspace of the integrable highest weight module corresponding to j, determined by a certain vacuum condition. These V_j are irreducible $sl(2,\mathbb{C})$-modules of dimension $2j+1$. Then the Virasoro algebra (centrally extended Lie algebra of Diff S^1) acts on each of these V_j. Given a vertex



satisfying $|j_1 - j_2| \le j \le j_1 + j_2$, $j_1 + j_2 + j \in \mathbb{Z}$, then $\exists$ a unique vertex operator for each j, mapping

$$V_j \otimes V_{j_1} \longrightarrow V_{j_2}.$$

Now inside Diff S^1 there is the important subgroup $SL(2,\mathbb{C})$ or $SU(1,1)$. While in 4 dimensions we do not have a corresponding infinite-dim. conformal group, twistor theory has $SL(4,\mathbb{C})$ or $SU(2,2)$. One can hope to turn the tables round and use some of the recent results on representations (e.g. the "discrete series") in twistor theory to get a handle on the proper twistorial form of vertex operators for a 4-dim. theory. The close relation between the two representation theories was expounded by RJB in a recent QFT seminar.

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