# Geometry on $\mathbb{CP}^1$ and the Virosoro Algebra

It is currently the rage to study a central extension of the Lie algebra of vector fields on the circle, the Virosoro algebra. The purpose of this note is to point out how the cocycle of this extension arises naturally from the geometry of $\mathbb{CP}_1$ under the action of $SU(1,1)$.

The starting point is $\mathbb{CP}_1$ with $\mathcal{U}^{\pm} = \{ z \mid |z| \gtrless 1 \}$, $S^1 = \{ z \mid |z| = 1 \} = N$ with $\Theta = \mathcal{O}(2) =$ sheaf of holomorphic vector fields and the complexification of the Lie algebra of analytic vector fields on the circle

$$V = \Gamma(N, \Theta) = \mathbb{C}\,\mathrm{Vect}(S^1)$$

(meaning sections near $N$). Mayer-Viëtoris gives $V = V^+ \oplus sl(2,\mathbb{C}) \oplus V^-$ where $V^{\pm} = \Gamma(\bar{\mathcal{U}}^{\pm}, \Theta)/sl(2,\mathbb{C})$ are in the (anti)-holomorphic discrete series for $SU(1,1)$ (cf decompositions of $E^8$ & $F^4$ into reps of $SO(16)$ & $SO(8)$, respectively). There is an exact sequence (Bernstein-Gelfand-Gelfand)

$$0 \longrightarrow sl(2,\mathbb{C}) \longrightarrow \Theta \xrightarrow{\ \delta^3\ } \Omega^{\otimes 2} \longrightarrow 0 \qquad (1)$$

$\uparrow$ sheaf of quadratic differentials

characterized by the fact that $\delta^3$ is left invariant by $SL(2,\mathbb{C})$. The corresponding long exact sequence gives

$$0 \longrightarrow sl(2,\mathbb{C}) \longrightarrow V \xrightarrow{\ \delta^3\ } \Gamma(N, \Omega^{\otimes 2}) \longrightarrow sl(2,\mathbb{C}) \longrightarrow 0$$

Put $L_k = z^{-k+1}\frac{d}{dz}$ so $[L_k, L_\ell] = (k-\ell)L_{k+\ell}$ and $\omega_k = z^{k-2}(dz)^2$ so $\delta^3 L_k = k(1-k^2)\omega_{-k}$. There is a natural pairing, for $\alpha \in V$, $\omega \in \Gamma(N, \Omega^{\otimes 2})$ given by

$$\langle \alpha, \omega \rangle = [\delta(\alpha \lrcorner\, \omega)] \in H^1(\mathbb{P}, \Omega^1) \cong \mathbb{C}$$

($\delta$ is the M-V connecting homomorphism). Ofcourse $\langle \alpha, \omega \rangle = \frac{1}{2\pi i}\int_{S^1} \alpha \lrcorner\, \omega$. This gives a skew symmetric 2-form on $V$ by

$$\Omega(\alpha, \beta) = \langle \alpha, \delta^3 \beta \rangle \qquad \text{so that } \Omega(L_k, L_n) = k(k^2-1)\delta_{k,-n}$$

trivial on $sl(2,\mathbb{C})$. This is (up to scale) the cocycle defining the Virosoro central extention by

$$[\alpha, \beta]_{\text{Virosoro}} = [\alpha, \beta]_V + \frac{1}{24}\Omega(\alpha,\beta)\,\underset{\sim}{c} \qquad \left(\text{physicists choose } \tfrac{1}{24}\right)$$

($\underset{\sim}{c}$ spans the centre). The Jacobi-identity for $[\,,]_{\text{Virosoro}}$ is equivalent to the cocycle condition on $\Omega$, ie.

$$\Omega([\alpha,\beta], \gamma) + \text{cyclic permutations} = 0$$

which follows because $[\alpha,\beta]\,\delta^3\gamma$ + cyclic perms $= d([\alpha,\beta]\,\delta^2\gamma$ + cyclic perms$)$ (in the langauge of Verma modules, there is a homomorphism of $[V(2)^{\otimes 3}]^* \to N(-2))^*$ factoring through $V(0)^*$).

Now it is still rather mysterious why the Virasoro cocycle should arise like this: usually, it comes from an embedding of Vect$(S^1)$ in a loop algebra & the Virasoro algebra is the pull back of the Kacs-Moody central extension of that:

$$\begin{array}{ccc} \text{Virasoro} & \longrightarrow & \hat{L}\,\mathfrak{sl}(2,\mathbb{C}) \\ \downarrow & & \downarrow \\ \text{Vect}\,S^1 & \xrightarrow[\text{(as vector fields on loops)}]{} & L\,\mathfrak{sl}(2,\mathbb{C}) \end{array} \qquad (2)$$

We can hunt around for a geometric explanation of this along the lines of what has just been done. A (complex) loop algebra is given by letting $\mathcal{O}(\mathfrak{sl}(2,\mathbb{C}))$ = $\mathfrak{sl}(2,\mathbb{C})$-valued functions on $\mathbb{P}_1$. Naturally, there is a differentially split sequence

$$0 \longrightarrow K \longrightarrow \mathcal{O}(\mathfrak{sl}(2,\mathbb{C})) \underset{\overset{D}{\longleftarrow}}{\longrightarrow} \Theta \longrightarrow 0$$

$D$ fails to be a homomorphism of Lie algebras — but only just: the obstruction is exactly $\alpha\,\delta^3\beta - \beta\,\delta^3\alpha \in \Omega^1$: nonetheless, if we define, for $\xi, \psi \in \Gamma(N, \mathcal{O}(\mathfrak{sl}(2,\mathbb{C}))) = \mathcal{L}$

$$\Omega_{\mathcal{L}}(\xi,\psi) = [\langle \xi \smile d\psi \rangle_{\text{Killing form}}] \in H^1(\mathbb{P}, \Omega^1) = \mathbb{C}$$
$$= \frac{1}{2\pi i}\int_{S^1} \langle \xi, d\psi \rangle_{\text{Killing form}}$$

then, remarkably, $\Omega_{\mathcal{L}}(D\alpha, D\beta) = \delta\Omega(\alpha,\beta)$ so $D$ would appear to be trying to understand (2). If one understood why this identity was true and its relation to (2) one might be able to construct representations of the Virasoro algebra from the Geometry of the projective line.

<u>Remark</u>: $\Gamma(N, \Omega^{\otimes 2}) \oplus \mathbb{C}$ is a (finite) dual of $\tilde{V}$ = Virasoro algebra. The infinitesimal co-adjoint actions of $L_k$ on $\Omega_{\alpha,\beta,j}$ (where $\Omega_{\alpha,\beta,j} = \beta\omega_j + \alpha\,\underset{\sim}{c}^*$) are given by

$$(\text{coad}\,L_n)\cdot\Omega_{\alpha,\beta,j} = \beta(j-2n)\omega_{j-n} - \alpha(1-n^2)n\,\omega_{-n}$$

Stabilizers of $\Omega_{\alpha,\beta,0}$ are as follows: always $L_0$ (& if this only, the orbit resulting is Diff $S^1/S^1$, studied by Rajeev & Bowick); if $2\beta + \alpha(1-n^2) = 0$ then $L_{\pm n}$ stabilize also and the orbit is Diff $S^1 / SL^{(n)}(2,\mathbb{R})$ (an $n$-fold covering of $SL(2,\mathbb{R}) = SU(1,1)$) (the real structure on $V$ is induced by the conjugate of the derivative of the antipodal map & given by $L_k \longrightarrow L_k^\dagger = -L_{-k}$: real vectors are $i(L_k + L_{-k})$ & $(L_k - L_{-k})$). This latter orbit is not evidently a "complex" manifold unless $n=1$. Observe also that Diff$(S^1)$ has no complexification, for if $\alpha = 0$ then $\omega_j$ is stabilized by something only if $j$ is even. On the other hand $\omega_{-1}, \omega_{-2}, \ldots$ lie on the same orbit under any complexification of Diff$(S^1)$ $\longrightarrow$ contradiction

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