The Geometry of Pure Spinors and Invariant differential operators in higher dimensions.

The Penrose transform for G/P can be used to construct invariant differential operators for forms on $\mathbb{C}S^{2n}$ = $2n$ dim Minkowski space (& hence, via Cartan connections, on all $2n$-dim conformal manifolds). One picks an appropriate homogeneous vector bundle on Z^{2n} = •—•…•<× = projective pure spinors for $so(2n+2,\mathbb{C})$ (the natural higher dimensional analogue of Twistor space) and calculates its Penrose transform.

The choice of such vector bundles (of zero infinitesmal character) is parameterized by the Hasse diagram of the parabolic q = •—•…•<× (a poset of the Weyl group of $so(2n+2,\mathbb{C})$). This is a directed graph whose nodes correspond to affine cells in Z^{2n} and whose edges give the attaching maps. There are certain specific cells in this specified as the top-dimensional cell in each Z^{2m} in the natural inclusions (given $so(m,\mathbb{C}) \hookrightarrow so(m+2,\mathbb{C})$)

$$\begin{array}{ccccccc} Z^2 & \hookrightarrow & Z^4 & \hookrightarrow & Z^6 & \hookrightarrow & \cdots\cdots \\ \shortparallel & & \shortparallel & & \shortparallel & & \\ \mathbb{C}\mathbb{P}_1 & & \mathbb{P}\mathbb{T} & & \text{6 dim'l quadric} & & \end{array} \qquad (1)$$

(Penrose & Rindler, Vol 2 indicates how to build higher spinors from lower ones)
From the point of view of Dynkin diagrams, (1) is just

 etc.

Let $w_k \in$ Weylgroup of $so(2n+2)$ be the element corresponding to such a cell in Z^{2n-2k}
The homogeneous bundle on Z^{2n} is $\mathcal{O}_q(w_k \cdot 0)$.

<u>Theorem</u>: $H^*(Z^{2n}, \mathcal{O}_q(w\cdot 0))$ is the total cohomology of

$$E_1^{p,q}: \quad \begin{array}{|cccccccc} & & & p=n-k+1 & & & & \\ & & & \downarrow & & & & \\ \Omega^k & \Omega^{k+1}\cdots & \Omega^n_+ & 0 & 0 & \cdots & 0 & \leftsquigarrow q = nk - k(k+1)/2 \\ 0 & 0\ \cdots & 0 & 0 & 0 & \cdots & 0 & \\ \vdots & \vdots\ \cdots & \vdots & \vdots & \vdots & \cdots & \vdots & \\ 0 & 0\ \cdots & 0 & \Omega^{2n-k+1} & \Omega^{2n-k+2} & \cdots & \Omega^{2n} & \leftsquigarrow q = n(k-1)-(k-1)k/2 \\ 0 & 0\ \cdots & 0 & 0 & 0 & \cdots & 0 & \\ \hline \end{array}$$

<u>Proof</u>: A calculation in the symmetric group on n-letters.

It follows that there are non-zero maps $\mathcal{D}: \Omega^{k-1} \longrightarrow \Omega^{2n-k+1}$ on $\mathbb{C}S^{2n}$ which are "non-standard" (ie, like $\square^2: \mathcal{O} \longrightarrow \Omega^4$ in four dimensions) (One has still to work hard to prove non-zeroness, here).

<u>Query</u>: can one use this to prove one has <u>all</u> invariant operators between forms?

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