A TWISTOR TRANSFORM FOR THE DISCRETE SERIES: The case of $SU(1,2)$.

§I. A twistorial alternative to the construction of representations via L^2-cohomology exists for the ladde representations of $SU(p,q)$. These form part of the analytic continuation of the discrete series for those groups. The crucial ingredient of the construction is the twistor transform for cohomology on $\mathbb{C}\mathbb{P}^n$:

$$\tilde{S}: H^s(\mathbb{P}^+, \mathcal{L}) \cong H^{s'}(\mathbb{P}^{*-}, \mathcal{L}^\vee)$$

where s and s' are the dimensions of the respective maximal compact subvarieties and $\mathcal{L}$ is a homogeneous line bundle with $\mathcal{L}^\vee = \mathcal{L}^* \otimes \Omega^{top}$. What follows is an example of how to do something similar for the discre series. The essential isomorphism holds for arbitrary semi-simple Lie groups. We will show how it works for $SU(1,2)$. The case of $SU(1,1)$ is a good exercise.

Let's set the notation:

$\mathbb{F} = \mathbb{F}_{12}(\mathbb{C}^3) = \{(L,P) \mid L = \text{line in } \mathbb{C}^3, P = \text{plane in } \mathbb{C}^3 \text{ and } L \subseteq P\}$.

$\mathbb{P} = \mathbb{C}\mathbb{P}^2 = \{\text{lines in } \mathbb{C}^3\}$

$\mathbb{F}_2 = G_2(\mathbb{C}^3) = \{\text{planes in } \mathbb{C}^3\} \cong \mathbb{P}^*$

Φ = Hermitian form of signature $(+--)$ which defines $SU(1,2)$

There is the double fibration:

$$\begin{tikzcd} & \mathbb{F} = \times\!-\!\times \arrow[dl] \arrow[dr] & \\ \mathbb{P} = \times\!-\!\bullet & & \mathbb{F}_2 = \bullet\!-\!\times \end{tikzcd}$$

Setting $G = SU(1,2)$, the G-orbits on these flag varieties are of the form:

$\mathbb{F}^{+,+-} = \{(L,P) \in \mathbb{F} \mid \Phi|_P \text{ has signature } (+-), \ \Phi|_L \text{ is positive definite}$

$\mathbb{P}^- = \{L \in \mathbb{P} \mid \Phi|_L \text{ is negative definite}\}$

$\mathbb{F}^{+-} = \mathbb{F}_2^{+-} = \{P \in \mathbb{F}_2 \mid \Phi_P \text{ has signature } (+-)\}$

and so on. For an explanation of notations involving Dynkin diagrams ($\times\!-\!\bullet$, $\times\!-\!\times$, $\bullet\!-\!\times$, $\bullet\!-\!\bullet$), see the manuscript by MGE & RJB.

§II The Penrose transform produces the isomorphisms:

$$H^1(\mathbb{P}^-, \overset{p}{\times}\!-\!-\!\overset{q}{\bullet}) \cong H^0(\mathbb{F}_2^{--}, \overset{q}{\bullet}\!-\!-\!\overset{-p-q-3}{\times}) \qquad p,q \geq 0$$

$$H^1(\mathbb{P}^-, \overset{-p}{\times}\!-\!-\!\overset{q}{\bullet}) \cong H^0(\mathbb{F}_2^{--}, \overset{q}{\bullet}\!-\!-\!\overset{p-q-3}{\times}) / \overset{q}{\bullet}\!-\!-\!\overset{p-q-3}{\bullet} \qquad q \geq 0,\ p \geq q+3$$

$$H^1(\mathbb{F}^{+-}, \overset{p}{\bullet}\!-\!-\!\overset{q}{\times}) \cong H^0(\mathbb{P}^+, \overset{-p-q-3}{\times}\!-\!-\!\overset{p}{\bullet}) \qquad p,q \geq 0$$

$$H^1(\mathbb{F}^{+-}, \overset{p}{\bullet}\!-\!-\!\overset{-q}{\times}) \cong H^0(\mathbb{P}^+, \overset{q-p-3}{\times}\!-\!-\!\overset{p}{\bullet}) / \overset{q-p-3}{\bullet}\!-\!-\!\overset{p}{\bullet} \qquad p \geq 0,\ q \geq p+3$$

Here $\overset{a}{\bullet}\!-\!-\!\overset{b}{\bullet}$ is a finite dimensional representation of $SL(3,\mathbb{C})$, which we will abbreviate to F. In some sense, factoring out by F is like factoring out the constants in other settings. There are the isomorphisms of the varieties:

$$\mathbb{P}^+ \cong \mathbb{F}_2^{*\,--}$$
$$\mathbb{P}^- \cong \mathbb{F}_2^{*\,+-}$$

effected by the map $z \longmapsto z^\perp = \{ \xi \in (\mathbb{C}^3)^* \mid \xi|_z = 0 \}$.

From these, we get the twistor transform, Mk. I:

(A) $$H^1(\mathbb{P}^-, \overset{-p}{\times}\!-\!-\!\overset{q}{\bullet}) \cong H^0(\mathbb{F}_2^{--}, \overset{q}{\bullet}\!-\!-\!\overset{p-q-3}{\times}) / F$$

$$\xrightarrow{\tilde{S}} \cong H^0(\mathbb{P}^{*+}, \overset{q}{\bullet}\!-\!-\!\overset{p-q-3}{\times}) / F$$

The last space is conjugate isomorphic to:

$$H^0(\mathbb{P}^+, \overset{p-q-3}{\bullet}\!-\!-\!\overset{q}{\times}) / F.$$

From now on, we will ignore the quotient by the finite dimensional subspace. Then from $\varphi, \psi \in H^1(\mathbb{P}^-, \overset{-p}{\times}\!-\!-\!\overset{q}{\bullet})$ form the cup product

$$\varphi \cup \overline{\tilde{S}\psi} \in H^1(\mathbb{P}^0, \overset{-q-3}{\times}\!-\!-\!\overset{2q}{\bullet}).$$

Follow this by the Mayer-Vietoris connecting map

$$\delta(\varphi \cup \overline{\tilde{S}\psi}) \in H^2(\mathbb{P}, \overset{-q-3}{\times}\!-\!-\!\overset{2q}{\bullet}).$$

When $q=0$, the vector bundle $\overset{-q-3}{\times}\!-\!-\!\overset{2q}{\bullet} = \overset{-3}{\times}\!-\!-\!\overset{0}{\bullet}$ is just $\Omega^2_{\mathbb{P}}$. Thus

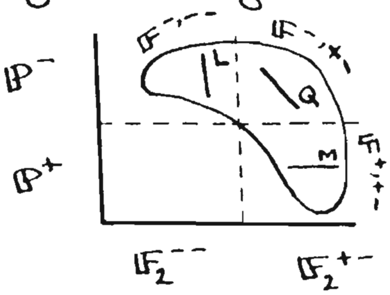
$$\delta(\varphi \cup \overline{\tilde{S}\psi}) := \varphi \cdot \overline{\psi} \in H^2(\mathbb{P}, \Omega^2) \cong \mathbb{C}.$$

(B) In a similar fashion:

$$\tilde{S}: H^1(\mathbb{P}^-, \overset{p}{\times}\!-\!-\!\overset{q}{\bullet}) \cong H^0(\mathbb{P}^{*+}, \overset{q}{\bullet}\!-\!-\!\overset{-p-q-3}{\times})$$

which is then conjugate isomorphic with $H^0(\mathbb{P}^+, \overset{-p-q-3}{\times}\!-\!-\!\overset{q}{\bullet})$. (No quotients by F occur here.) Again, elements pair into $H^2(\mathbb{P}, \overset{-q-3}{\times}\!-\!-\!\overset{2q}{\bullet})$ which is isomorphic with $\mathbb{C}$ for $q=0$. The bundles $\overset{p}{\times}\!-\!-\!\overset{0}{\bullet}$ are the line bundles on $\mathbb{P}$.

§III Using part II, we shall give a twistor transform for discrete series representations. In general, the discrete series occur in the cohomology of (homogeneous) line bundles over the open G-orbits in the full flag variety (in this case, $\mathbb{F}_{12}$). The line bundles should satisfy a negativity requirement (similar to anti-dominance for lowest weights in finite dimensional representations - especially as treated in the Borel-Weil theorem). The flag variety naturally imbeds into the product $\mathbb{F}_2 \times \mathbb{P}$



The open G-orbits in $\mathbb{F}_{12}$ are $\mathbb{F}^{-,--}$, $\mathbb{F}^{-,+-}$ and $\mathbb{F}^{+,+-}$. L, Q and M represent their respective maximal compact subvarieties (which are $\mathbb{CP}^1$'s). Under the natural projections

$$\begin{array}{ccc} & \mathbb{F}_2 \times \mathbb{P} & \\ \swarrow^{\pi_1} & & \searrow^{\pi_2} \\ \mathbb{F}_2 & & \mathbb{P} \end{array}$$

$\mathbb{F}^{-,--}$ projects to $\mathbb{F}_2^{--}$ with fiber $\mathbb{CP}^1$ and $\mathbb{F}^{+,+-}$ projects to $\mathbb{P}^+$ with fiber $\mathbb{CP}^1$. $\mathbb{F}^{-,+-}$ is a little tougher to work with, hence interesting. However we will avoid this case. For any subset Y of $\mathbb{P}$, $\mathbb{F}_2$ or $\mathbb{F}$, let $|Y|$ denote its closure and $\overline{Y}$ its complex conjugate. Notice:

$$\overline{\mathbb{F}^{-,--}} = \mathbb{F}^{+,+-} \quad \text{and} \quad \overline{\mathbb{F}^{-,+-}} = \mathbb{F}^{-,+-}.$$

To produce discrete series representations, let's consider

$$H^1(|\mathbb{F}^{-,--}|, \overset{-p}{\times}\!\!-\!\!\!-\!\!\overset{-q}{\times}) \quad \text{with } q \geq 0 \text{ and } p \geq 2.$$

Using the fibration π_2 and the earlier isomorphisms we have:

$$H^0(|\mathbb{F}^{-,--}|, \overset{-p}{\times}\!\!-\!\!\!-\!\!\overset{-q}{\times}) \cong H^0(|\mathbb{F}_2^{--}|, \overset{p-2}{\bullet}\!\!-\!\!\!-\!\!\overset{-p-q+1}{\times})$$

$$\cong H^1(|\mathbb{P}^-|, \overset{q-2}{\times}\!\!-\!\!\!-\!\!\overset{p-2}{\bullet})$$

Using π_1:

$$H^1(|\mathbb{P}^-|, \overset{q-2}{\times}\!\!-\!\!\!-\!\!\overset{p-2}{\bullet}) \cong H^1(|\mathbb{F} - \mathbb{F}^{+,+-}|, \overset{q-2}{\times}\!\!-\!\!\!-\!\!\overset{p-2}{\times}).$$

Putting it all together produces the twistor transform, Mk. II:

$$\boxed{\tilde{S} : H^1(|\mathbb{F}^{-,--}|, \overset{-p}{\times}\!\!-\!\!\!-\!\!\overset{-q}{\times}) \cong H^1(|\mathbb{F} - \mathbb{F}^{+,+-}|, \overset{q-2}{\times}\!\!-\!\!\!-\!\!\overset{p-2}{\times})}$$

whence, for $\varphi, \psi \in H^1(|\mathbb{F}^{-,--}|, \overset{-p}{x}\!\!-\!\!\overset{-q}{x})$ we have $\overline{\tilde{\mathfrak{z}}\psi} \in H^1(|\mathbb{F} - \mathbb{F}^{-,--}|, \overset{p-2}{x}\!\!-\!\!\overset{q-2}{x})$. Applying the Mayer-Vietoris connecting map to the cup product gives

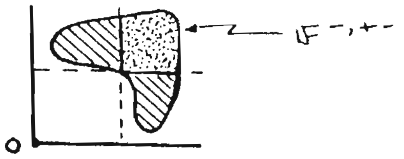
$$\delta(\varphi \cup \overline{\tilde{\mathfrak{z}}\psi}) \in H^3(\mathbb{F}_{12}, \overset{-2}{x}\!\!-\!\!\overset{-2}{x}) = H^3(\mathbb{F}, \Omega^3) \cong \mathbb{C}.$$

This produces the G-invariant Hermitian pairing necessary for unitarizing the discrete series. Similar pairings exist for the cohomology groups on $\mathbb{F}^{+,+-}$ and $\mathbb{F}^{-,+-}$.

§ IV NOTES:

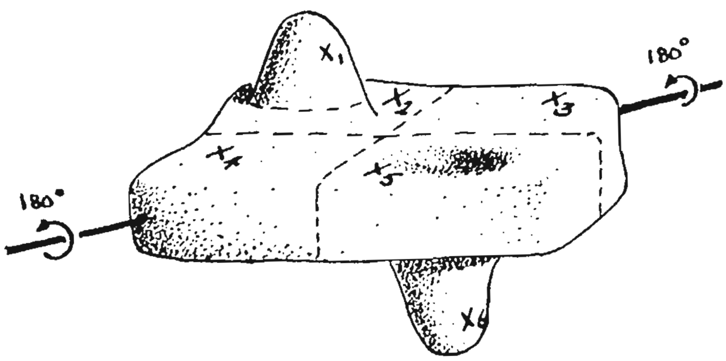
1) The quotient by F occurs only for 'small' groups such as SU(1,1) and SU(1,2).

2) The picture of the isomorphism for $\mathbb{F}^{-,+-}$ is that cohomology on $|\mathbb{F}^{-,+-}|$ is isomorphic to cohomology on $|\mathbb{F} - \mathbb{F}^{-,+-}|$:



Cohomology on the 'dots' $\cong$ cohomology on the 'lines'.

3) For SU(2,2) the picture of $\mathbb{F}_{123}$ in $\mathbb{P} \times \mathbb{F}_2 \times \mathbb{F}_3$ is:



The permutation of the orbits effected by conjugation can be visualized as rotation by 180° about an axis through the orbits X_3 and X_4. (Conjugation on $\mathbb{F}_{12}$ has a similar interpretation) Thus $X_1 \leftrightarrow X_6$ $X_2 \leftrightarrow X_5$ $X_3 \circlearrowleft$ and $X_4 \circlearrowleft$

Again cohomology on the orbit X_i is related to cohomology on the complement of its conjugate, $\overline{X}_i$:

$$H^s(|X_i|, \mathcal{X}) \cong H^d(|X - \overline{X}_i|, \mathcal{X}^\vee).$$

where the degrees are just right to get a pairing into $H^6(\mathbb{F}_{123}, \Omega^{top}) \cong \mathbb{C}$.

4) The general isomorphism (arbitrary groups) can be proved on the level of elementary states (a.k.a. K-types) by comparing a calculation of local cohomology [EJB] with a calculation of formal neighborhoods as in Schmid's thesis. For the isomorphism as representations,

the machinery of $\mathcal{D}$-modules is needed. The twistor transform is then interpretted as a relation between two different types of direct image modules.

Many thanks to RJB.

Mike Eastwood and Ed Dunne.

References:

[RJB & MGE] The Penrose Transform: its interaction with representation theory.

[RJB] Local Cohomology, elementary states and evaluation, TN #22.

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A generalised Kerr–Robinson theorem

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Abstract. The Kerr and Robinson theorems in four-dimensional spacetime together provide the general null solution of Maxwell's equations. Robinson's theorem reduces the problem to that of obtaining certain null foliations. The Kerr theorem shows how to represent such foliations in terms of analytic varieties in complex projective 3-space. In this paper we generalise these results to spinor fields of higher valence in spacetimes of arbitrary even dimension. We first review the theory of spinors and twistors for these higher dimensions. We define the appropriate generalisations of Maxwell's equations, and null solutions thereof. It is then proved that the Kerr and Robinson theorems generalise to all even dimensions. We discuss various applications, examples and further generalisations. The generalised Robinson theorem can be seen to extend to curved spaces which admit such null foliations. In the case of Euclidean reality conditions, the generalised Kerr theorem determines all complex structures compatible with the flat metric in terms of freely specified complex analytic varieties in twistor space. Interpretations of the generalised Kerr theorem are also provided for Lorentzian and ultrahyperbolic signatures.