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A conformally invariant connection and the space of leaves of a shear free congruence

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Introduction

This is a report on work in progress, studying the structure of the complex surface which is the space of leaves of a (complexified) shear free congruence. I will show below that in conformal vacuum space-times, the surface has the first formal neighbourhood of an embedding in a complex three manifold (which in the flat space would be dual projective Twistor space).

In order to describe this structure, I will first show that a conformal complex space-time with two spinor fields has a natural conformally invariant connection, which is essentially given by R.P.'s 'conformally invariant edth and thorn operators'. This construction seems to have some geometric interest in its own right.

It is hoped that these these structures will help to explain the separation of various equations in the Kerr metric, and there may be other applications.

The conformally invariant connection

Let $\mathcal{M}$ be a complex conformal space-time, with two independent spinor fields o^A and ι^A, defined up to scale. Equivalently we have a splitting

$$\mathcal{O}^A = O \oplus I \tag{1}$$

of the spin bundle.

Assume also that we are given an identification of the primed and unprimed conformal weights

$$[1] \stackrel{def}{=} \mathcal{O}_{[AB]} \cong \mathcal{O}_{[A'B']}$$

This is equivalent to allowing conformal transformations only of the form

$$\epsilon_{AB} \mapsto \Omega\epsilon_{AB} \qquad\qquad \epsilon_{A'B'} \mapsto \Omega\epsilon_{A'B'}$$

which is a natural condition if $\mathcal{M}$ is the complexification of a real space-time.

Given a metric in the conformal class, the splitting in equation 1 allows us to define a one form

$$Q_a := -2o^{(B}\iota^{C)}\partial_{A'B}(o_{(A}\iota_{C)}) = \rho' l_a + \rho n_a - \tau' m_a - \tau\bar{m}_a$$

where ∂_a is the metric connection, and we adopt the convention that $o_A\iota^A = 1$ whenever a particular metric has been chosen. Under conformal transformatiom

$$Q_a \mapsto Q_a - \Upsilon_a \qquad \text{where} \qquad \Upsilon_a = \Omega^{-1}\partial_a\Omega \tag{2}$$

The significance of Q_a is that it enables us to split the *Local Twistor* bundle as a direct sum.

Recall the Local Twistor exact sequence

$$\begin{array}{ccccccccc} 0 \longrightarrow & \mathcal{O}_{A'} & \longrightarrow & \mathcal{O}^{\alpha} & \longrightarrow & \mathcal{O}^A & \longrightarrow 0 \\ & \pi_{A'} & \longmapsto & (0, \pi_{A'}) & & & \\ & & & (\omega^A, \pi_{A'}) & \longmapsto & \omega^A & \end{array}$$

and the conformal transformation rule

$$\omega^A \mapsto \omega^A \qquad\qquad \pi_{A'} \mapsto \pi_{A'} + i\Upsilon_a\omega^A$$

If we set

$$\alpha_{A'} = \pi_{A'} + iQ_a\omega^A$$

then from equation 2 there is a conformally invariant splitting

$$\begin{array}{ccc} \mathcal{O}^{\alpha} & \stackrel{\simeq}{\longrightarrow} & \mathcal{O}^A \oplus \mathcal{O}_{A'} \\ (\omega^A, \pi_{A'}) & \longmapsto & \omega^A \oplus \alpha_{A'} \end{array} \tag{3}$$

of $\mathcal{O}^\alpha$ and I will use the 'split co-ordinates' $(\omega^A, \alpha_{A'})$ henceforth.

The *Local Twistor connection* splits to give connections, which I will denote by ∇_a, on the various spin bundles. A brief calculation shows these to be

$$\begin{array}{ll} \mathcal{O}^A & : \nabla_b\mu^A = \partial_b\mu^A + \epsilon_B{}^A Q_{CB'}\mu^C \\ \mathcal{O}^{A'} & : \nabla_b\mu^{A'} = \partial_b\mu^{A'} + \epsilon_{B'}^{A'} Q_{BC'}\mu^{C'} \\ \mathcal{O}_A & : \nabla_b\mu_A = \partial_b\mu_A - Q_{AB'}\mu_B \\ \mathcal{O}_{A'} & : \nabla_b\mu_{A'} = \partial_b\mu_{A'} - Q_{BA'}\mu_{B'} \\ \mathcal{O}_{[AC]} & : \nabla_b\nu_{AC} = \partial_b\nu_{AC} - Q_b\nu_{AC} \end{array}$$

If $Z^\alpha = (\omega^A, \alpha_{A'})$ is a local twistor, we can write the Local Twistor connection as

$$\nabla_b Z^\alpha = (\nabla_b\omega^A + i\epsilon_B{}^A\alpha_{B'}, \nabla_b\alpha_{A'} + iD_{ab}\omega^A) \tag{4}$$

where D_{ab} is a conformally invariant modification of P_{ab} defined by

$$D_{ab} = P_{ab} - \partial_b Q_a + Q_{AB'}Q_{BA'}$$

The splitting in equation 1 allows us to define the bundles

$$\langle -r', -r\rangle := O^{r'} \otimes I^r$$

(note that $\langle 1,1\rangle = [1]$). The connection ∇_a can be projected on to these. For example, if λ^A is a section of $\langle -1, 0\rangle$, so that $\lambda^A o_A = 0$,

$$\lambda^A \longmapsto -o^A\iota_C\nabla_b\lambda^C$$

is a connection, and its components are given by 'conformally invariant edth and thorn', in just the same way as the same expression with the metric connection ∂_b replacing ∇_b has components that can be computed with ordinary edth and thorn.

Since ∇_a agrees with ∂_a if you form any of the well known conformally invariant parts of the metric connection, there is scope here for producing a complete 'conformally invariant G.H.P. formalism'. The expressions which arise as curvatures when one commutes conformal edths and thorns are components of D_{ab}. The geometrical significance of these connections will be discussed in a later section.

Shear free congruences in Minkowski space

Before starting on the general case, I will review the situation in flat space-time. In real Minkowski space, a shear free congruence of null geodesics (hereafter SFR) is given by a spinor field satisfying

$$o^A o^B \partial_a o_B = 0 \tag{5}$$

If o_A is analytic, it can be complexified, and it then determines a distribution of β-planes. This distribution is integrable, and so gives a foliation of Minkowski space by complex surfaces precisely when o_A is shear free. The space of leaves S of this foliation is the hypersurface in dual projective Twistor space $\mathbf{P}^*$, which describes the congruence, according to Kerr's Theorem.

The surface S inherits some structure from its embedding, in particular there is the tangent bundle of $\mathbf{P}^*$ which sits in the *normal bundle sequence* and the restrictions of the line bundles $\mathcal{O}(n)$. The analysis in the accompanying article in this T.N. shows how massless fields of various orders along the congruence are isomorphic to sections of sheaves on S. I will now describe how this generalises to curved space.

SFRs in curved space-times

In a general space-time, an SFR is still given by a solution of equation 5, and gives a foliation in the complexification. The space of leaves still defines a complex surface S, but there is in general no Twistor space in which it is embedded.

The SFR defines a Maxwell field, which in Minkowski space is the Ward transform of the line bundle defined by S considered as a divisor. This follows from the fact that equation 5 is equivalent to the existence of a one form Φ_a with

$$\partial_{A'(A} o_{B)} = \Phi_{A'(A} o_{B)}$$

and it is easy to see that Φ_a has precisely the freedom to be the potential for a Maxwell field. The left handed part $\phi_{AB} = \partial_{A'(A}\Phi^{A'}_{B)}$ satisfies $\Psi_{ABCD}o^D = -\phi_{(AB}o_{C)}$, and so vanishes as expected in a conformally flat space- time[1].

[1]An SFR is thus a charged Twistor coupled to its own canonically defined Maxwell field

The structures I shall describe on S only exist under certain conditions. In particular, I will say that the SFR o_A in the space-time $\mathcal{M}$ satisfies the *Goldberg–Sachs condition* (hereafter GS) if

$$o^A o^B o^C \Psi_{ABCD} = 0$$

We assume the GS condition holds henceforth, since no significant part of the structure on S seems to exist otherwise. The Goldberg–Sachs Theorem implies that the GS condition is equivalent to $o^A o^B o^C \partial^D_{D'} \Psi_{ABCD} = 0$ and it is therefore satisfied by all conformally vacuum space-times.

To construct bundles on S, we make use of ∇_a, the conformally invariant connection. First choose a spinor direction ι_A to complement the SFR o_A, and deduce from the SFR and GS conditions that on all the bundles $\langle r', r\rangle$ and $\mathcal{O}^{A'}$, the part $o^A\nabla_a$ of the connection that differentiates up the leaves of the foliation is both independent of the choice of ι_A and flat.[2]

We can thus define line bundles $\langle r', r\rangle_S$ and a rank two vector bundle $\mathcal{O}(S)^{A'}$ over S, whose sections are ***by definition*** sections of the corresponding bundle on $\mathcal{M}$ with vanishing conformal derivitive up the foliation.

The dual Local Twistor bundle also defines a vector bundle on S. We have an injection of the spinors proportional to o_A into $\mathcal{O}_\alpha$

$$0 \longrightarrow \langle 0,1\rangle \longrightarrow \mathcal{O}_\alpha \longrightarrow E \longrightarrow 0$$

defining the quotient E. The part $o^A\nabla_a$ of the Local Twistor connection preserves $\langle 0,1\rangle$ and hence is well defined on E. Furthermore, it is flat on the leaves and so defines a rank three vector bundle $\mathcal{E}$ on S.

Sections of E can be realised as spinor fields $\xi^{A'}$ satisfying a tangential Twistor equation[3]

$$o^A \nabla_A{}^{(A'}\xi^{B')} = 0$$

and given that sections of $\mathcal{O}(S)^{A'}$ are spinor fields satisfying

$$o^A \nabla_{AA'}\xi^{B'} = 0$$

[2]It is helpful to note that $o^A Q_a$ is independent of ι_A if o^A is SFR.

[3]to see this, note that GS and SFR imply $o^A o^B D_{ab} = 0$ and use the conjugate version of equation 4. When writing down the splitting and connection on the dual Local Twistors, simply write down the conjugate *pretending that D_{ab} and Q_a are real.*

we get an injection $\mathcal{O}(S)^{A'} \to \mathcal{E}$ which extends to give a short exact sequence

$$0 \longrightarrow \mathcal{O}(S)^{A'} \longrightarrow \mathcal{E} \longrightarrow \langle 1,0\rangle \longrightarrow 0$$

given, in terms of equations, by

$$o^A \nabla_{AA'} \xi^{B'} = 0 \quad \longmapsto \quad o^A \nabla_A{}^{(A'} \xi^{B')} = 0 \quad \longmapsto \quad \begin{pmatrix} o^A o^B \nabla_{BB'} \eta_A = 0 \\ \iota^A \eta_A = 0 \end{pmatrix}$$

$$\xi^{A'} \quad \longmapsto \quad \iota_A o^B \nabla_{BB'} \xi^{B'}$$

If $\mu^{A'}$ is a section of $\mathcal{O}^{A'}\langle 0,-1\rangle$, a calculation reveals that the condition $o^A \nabla_a \mu^{B'} = 0$ is what is required[4] to make $\iota^A \mu^{A'}$ a connecting vector to a nearby leaf of the foliation. Thus, $\mathcal{O}(S)^{A'}\langle 0,-1\rangle_S$ can be identified with the tangent bundle $T(S)$ of S. The exact sequence above can be tensored through by $\langle 0,-1\rangle_S$ to give what in flat space would be the *normal bundle sequence* of S

$$0 \longrightarrow T(S) \longrightarrow \mathcal{E}\langle 0,-1\rangle_S \longrightarrow \langle 1,-1\rangle_S \longrightarrow 0$$

If one is given a hypersurface in a complex manifold, then knowing the normal bundle sequence is equivalent to knowing the *first formal neighbourhood* of the embedding. I will now briefly describe how one can realise the first formal neighbourhood of an embedding of S directly.

The spinor field o_A defines a natural embedding of the space-time $\mathcal{M}$ in the *projective spin bundle* $\mathbf{P}\mathcal{O}_A$. Now realise S by choosing a two-surface $\tilde{S}$ transverse to the foliation, and note that $\tilde{S}$ has a natural embedding in the restriction of $\mathbf{P}\mathcal{O}_A$. The *first formal neighbourhood* of this embedding is independent of the choice of $\tilde{S}$, and so defines a first formal neighbourhood sheaf $\mathcal{O}^{(1)}$ on S.

In slightly more detail; recall that $\mathbf{P}\mathcal{O}_A$ has a naturally defined differential operator $\pi^A \partial_a$ which defines a two-plane distribution, the integral surfaces of which (if it has any) are lifts of β-surfaces.

When o_A is an SFR, there is a two complex parameter family of β-surfaces parametrised by S, and functions f on $\mathbf{P}\mathcal{O}_A$ which obey $\pi^A \partial_a f = 0$ *on the lift of* $\mathcal{M}$ are precisely functions on S.

[4] The connection here is the tensor product of the conformally invariant ones on the factors

A calculation shows that, given the GS condition, there are two functions of two complex variables worth of functions g on $\mathbf{P}\mathcal{O}_A$ that obey $\pi^A \partial_a g = 0$ *to first order in a neighbourhood of the lift of* $\mathcal{M}$. These form the formal neighbourhood sheaf $\mathcal{O}^{(1)}$ on S.

In terms of the conformally invariant connections, a function on the first formal neighbourhood of the lift of $\mathcal{M}$ can be written

$$g(x, \pi_A) = f(x) + \iota^A \chi_A{}^B \pi_B \qquad \text{where} \qquad o^A \chi_A{}^B = 0 = \chi_A{}^B o_B$$

If the spinor field $\chi_A{}^B$ satisfies

$$\nabla_{BA'} \chi_A{}^B = \nabla_{AA'} f$$

then it defines a section of $\mathcal{O}^{(1)}$.

Massless fields

One result of this analysis is a minor generalisation of Robinson's Theorem, which states that if o_A is an SFR, then, for each helicity, there are precisely one holomorphic function of two complex variables worth of left handed massless fields null along it. If the field has n indicees, then remembering that it has conformal weight -1, it is easy to check that these fields are in one to one correspondence with sections over S of $\langle 1, n+1 \rangle_S$.

In my accompanying article I show how *in flat space* fields of various orders along o_A correspond to sections of sheaves over S. Provided, as usual, that the GS condition holds, it turns out that sections of the formal neighbourhood sheaf $\mathcal{O}^{(1)} \otimes \langle 1, 1 \rangle_S$ on S do give left handed Maxwell fields which have a principal null direction along the congruence. Thus there are two holomorphic functions of two complex variables worth of such things, just as in the flat case.

Apart from that case however, more severe curvature restrictions appear. To get three functions worth of order three Maxwell fields one requires $o^A o^B \Psi_{ABCD} = 0$ in which case it seems that S has a second formal neighbourhood sheaf.

Killing spinors

Suppose $\mathcal{M}$ admits a *Killing spinor*, and choose o_A and ι_A to be along its principal null directions. The Killing spinor equation

$$\partial_{A'}{}^{(A}\omega^{BC)} = 0$$

then implies that both o_A and ι_A are SFRs. The remaining parts of the equation reduce to solving $\nabla_a\omega = 0$ where ω is a section of $\langle 1,1\rangle$. This is only possible if the conformally invariant connection on $\langle 1,1\rangle$ is flat, which implies

$$\partial_{[a}Q_{b]} = 0$$

This has a number of consequences. Firstly, it provides an isomorphism $\langle 1,1\rangle \cong \langle 0,0\rangle$ which carries over to S, thereby giving a natural trivialisation of $\langle 1,1\rangle_S$. Secondly, the fact that Q_a is closed means that locally it is exact, and equation 2 shows that it can thus be made to vanish by a conformal transformation. In the special metric thus constructed, *all* the curvature information is contained in the single line bundle $\langle 1,0\rangle$ and its (conformally invariant) connection[5]. Further work is in progress on all this, since it seems likely that, combined with the ideas in the next section, it will be possible to explain the separation of various differential equations in the Kerr solution.

Geometrical significance

To finish, I will mention some ideas due to R.P. and K.P.T. which I have just started to follow up in collaberation with M.A.S. The conformally invariant connection constructed above is an example of a unique connection determined by a geometrical structure, and the structure one has (in the complex space- time) seems to be that which would be obtained on the complexification of a real four manifold X with an almost complex structure $J_a{}^b$ and a compatible conformal Hermitian metric. The eigenspaces of $J_a{}^b$ are the two-plane distributions defined by o_A and ι_A so that

$$J_a{}^b = i(o_A\iota^B + \iota_A o^B)\epsilon_{A'}{}^{B'}$$

The almost complex structure will be integrable when both o_A and ι_A are SFRs. Further, the suggestion is that the existence of a Killing spinor

[5] c. f. B.P.J. in Proc. Roy. Soc. A392 p323–341 (1984)

is equivalent to the Kähler condition on the Hermitian metric. This seems very likely since something very similar has been given by Flaherty[6], whose view-point is somewhat different.

I would like to thank M.A.S., R.P., and K.P.T. for discussions and suggestions.

[6] Hermitian and Kählerian geometry in relativity. Lecure Notes in Physics 46 (1976)