Relative cohomology power series, Robinson's Theorem and multipole expansions

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Introduction

In my original articles on the Twistor description of fields with sources on a world-line[1] I gave some expressions for "multipoles" based on a world-line. In this note, I will show how a first cohomology class, relative to a hypersurface, can be expanded in a sort of "power series", which seems to be the Twistor version of the multipole expansion. The power series also gives a precise "abstract nonsense" version of the Twistor description of algebraically special fields.

The relative cohomology power series

Let S be a hypersurface in a complex manifold X, and let $\mathcal{F}$ be a locally free sheaf of $\mathcal{O}_X$ modules on X. The *relative cohomology group* $H^1_S(X, \mathcal{F})$ can be described by a relative Čech cocycle, but a good intuitive picture is as follows: Choose an open cover U_i of a neighbourhood of S in X; then a representative is given by a set f_i of sections of $\mathcal{F}$ over U_i that 'blow up' on S, with the restriction that $f_i - f_j$ is holomorphic on *all* of $U_i \cap U_j$. The freedom in each f_i is the addition of a holomorphic section of $\mathcal{F}$.

[1]T.N. 14,15 and Proc. Roy. Soc. A397 143–155 (1985)

Now let g_i be *defining functions* for S, then one might try and expand the relative class defined by the f_i as a power series

$$f_i = \frac{f_i^{(1)}}{g_i} + \frac{f_i^{(2)}}{g_i^2} + \cdots \frac{f_i^{(n)}}{g_i^n} + \cdots \tag{1}$$

To understand this we need the *divisor bundle* L of S, which is defined to be the line bundle with transition functions g_i/g_j on $U_i \cap U_j$. The functions g_i then give a *distinguished section* s of L which has a simple zero on S. The section s gives us a map

$$s^k : \mathcal{F} \longrightarrow \mathcal{F} \otimes L^k$$

which induces a map on the relative cohomology.

Definition 1 *The k-th order relative cohomology $H^1_S(X, \mathcal{F}; k)$ is defined by the exactness of*

$$0 \longrightarrow H^1_S(X, \mathcal{F}; k) \longrightarrow H^1_S(X, \mathcal{F}) \xrightarrow{\times s^k} H^1_S(X, \mathcal{F} \otimes L^k)$$

The k-th order cohomology is thus the part which has a pole of order k or less on S, and it therefore corresponds to the first k terms in equation 1 above.

If $\mathcal{E}$ is a sheaf on X, and $\mathcal{I}^{(p)}\mathcal{E}$ is the ideal of sections of $\mathcal{E}$ which vanish to p-th order on S we can define the *p-th formal neighbourhood sheaf* $(\mathcal{E})^{(p)}$ by the short exact sequence

$$0 \longrightarrow \mathcal{I}^{(p+1)}\mathcal{E} \longrightarrow \mathcal{E} \longrightarrow (\mathcal{E})^{(p)} \longrightarrow 0 \tag{2}$$

so that $(\mathcal{E})^{(0)}$ is just $\mathcal{E}$ restricted to S.

Lemma 1 *There is a natural isomorphism*

$$H^1_S(X, \mathcal{F}; k) \cong \Gamma(S, (\mathcal{F} \otimes L^k)^{(k-1)})$$

The proof is simply to observe that in equation 1 above, the $f_i^{(k)}$ must give a section of $\mathcal{F} \otimes L^k$ with the freedom as given by equation 2.

Thus we have strictly a *filtration* of the relative cohomology (rather than an infinite direct sum), with the quotient at each stage given by the exact sequence

$$0 \longrightarrow \Gamma(S, (\mathcal{F} \otimes L^{k-1})^{(k-2)}) \xrightarrow{\times s} \Gamma(S, (\mathcal{F} \otimes L^k)^{(k-1)} \longrightarrow \Gamma(S, \mathcal{F} \otimes L^k) \longrightarrow 0$$

Algebraically special fields

The above analysis can be applied when S is a hypersurface in a region X in projective Twistor space, corresponding to a shear free congruence. We can define cohomology of order k on S just as for the relative case, and we will say that a right handed massless field is of order k on the congruence if its Twistor function is in $H^1(X, \mathcal{O}(-n-2); k)$. Thus, order 1 means null, order n means the field has a pnd. along the congruence, and higher orders correspond to certain differential relations between the field and the congruence.

If one writes down the commutative diagram whose rows are the relative cohomology sequences, and whose columns are induced by $s^k : \mathcal{F} \to \mathcal{F} \otimes L^k$, it is easy to see that if $H^1(X, \mathcal{F}) = 0$ then there is an exact sequence

$$0 \longrightarrow \frac{\Gamma(X, \mathcal{F} \otimes L^k)}{\Gamma(X, \mathcal{F})} \longrightarrow H^1_S(X, \mathcal{F}; k) \longrightarrow H^1(X, \mathcal{F}; k) \longrightarrow 0$$

Since L has the section s which has a simple zero on S, which intersects every line in X exactly once, we can write $L = M(1)$ where M is a line bundle trivial on every line in X[2]. Thus if $k < n + 2$

$$\Gamma(X, L^k(-n-2)) = \Gamma(X, M^k(k-n-2)) = 0$$

and so

$$H^1_S(X, \mathcal{O}(-n-2); k) \cong H^1(X, \mathcal{O}(-n-2); k); \quad k < n+2$$

The result of all this is a statement of the (generalised) flat space Robinson Theorem: The space of helicity $n/2$ right handed massless fields of order k ($k < n+2$) along the congruence is isomorphic to $\Gamma(S, (L^k(-n-2))^{(k-1)})$. This is precisely the 'k holomorphic functions of two complex variables' described by R.P. and W.R. in SS-T. (Vol. 2 p. 206).

The particular case where $k = 1$ and $n = 2$ was examined by M.G.E. (T.N.20 p. 31). We get that these fields are given by sections of $L(-4)$ over S, but $\mathcal{O}(-4) = \Omega^3$ and L restricted to S is just the normal bundle. Thus

[2] M is the Ward bundle of the 'Maxwell field of the congruence' — see my accompanying article

$L(-4) = \Omega_S^2$, we get an isomorphism of the null right handed Maxwell fields with holomorphic 2-forms on S.

The null Maxwell fields inject into the order 2 fields, and give a quotient sheaf

$$0 \longrightarrow \Gamma(S, L(-4)) \longrightarrow \Gamma(S, (L^2(-4))^{(1)}) \longrightarrow \Gamma(S, L^2(-4)) \longrightarrow 0$$

The quotient corresponds in space-time to neutrino fields of order 1, coupled to the Maxwell field of the congruence. The map onto this group is 'helicity lowering', where the congruence is regarded as a charged Twistor.

Multipole expansions

If S is the *ruled surface* corresponding to a world-line in Minkowski space, the first relative cohomology describes massess fields with sources on the world-line[3]. We can use the analysis given above to get a filtration of these fields.

It seems that the first terms (eg. order 2 for right handed Maxwell and order 3 for right handed gravity) give the fields with non-vanishing 'charges',and the remainder give an expansion in 'multipoles' where, for example, a 2^p- pole for a right handed helicity $n/2$ field is given by

$$\phi_{\underbrace{A'\ldots K'}_{n}} = \oint \sigma^{\overbrace{A\ldots N}^{n+p}\overbrace{P\ldots S}^{p}} \dot{y}_P^{L'} \ldots \dot{y}_S^{N'} \nabla_{AA'} \ldots \nabla_{NN'} \frac{ds}{(x - y(s))^2}$$

where $\sigma^{A\ldots S}$ is a totally symmetric spinor function of s, the proper time along the world-line $y^a(s)$. Under conformal transformations, a 2^p-pole gets mixed with lower ordered terms, which is what one might expect given that the Twistor space expansion is not a direct sum.

There are still many details to be tidied up here, and further work is in progress.

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[3]and the Maxwell field of the congruence is the left handed part of the field of a unit charge on the world-line