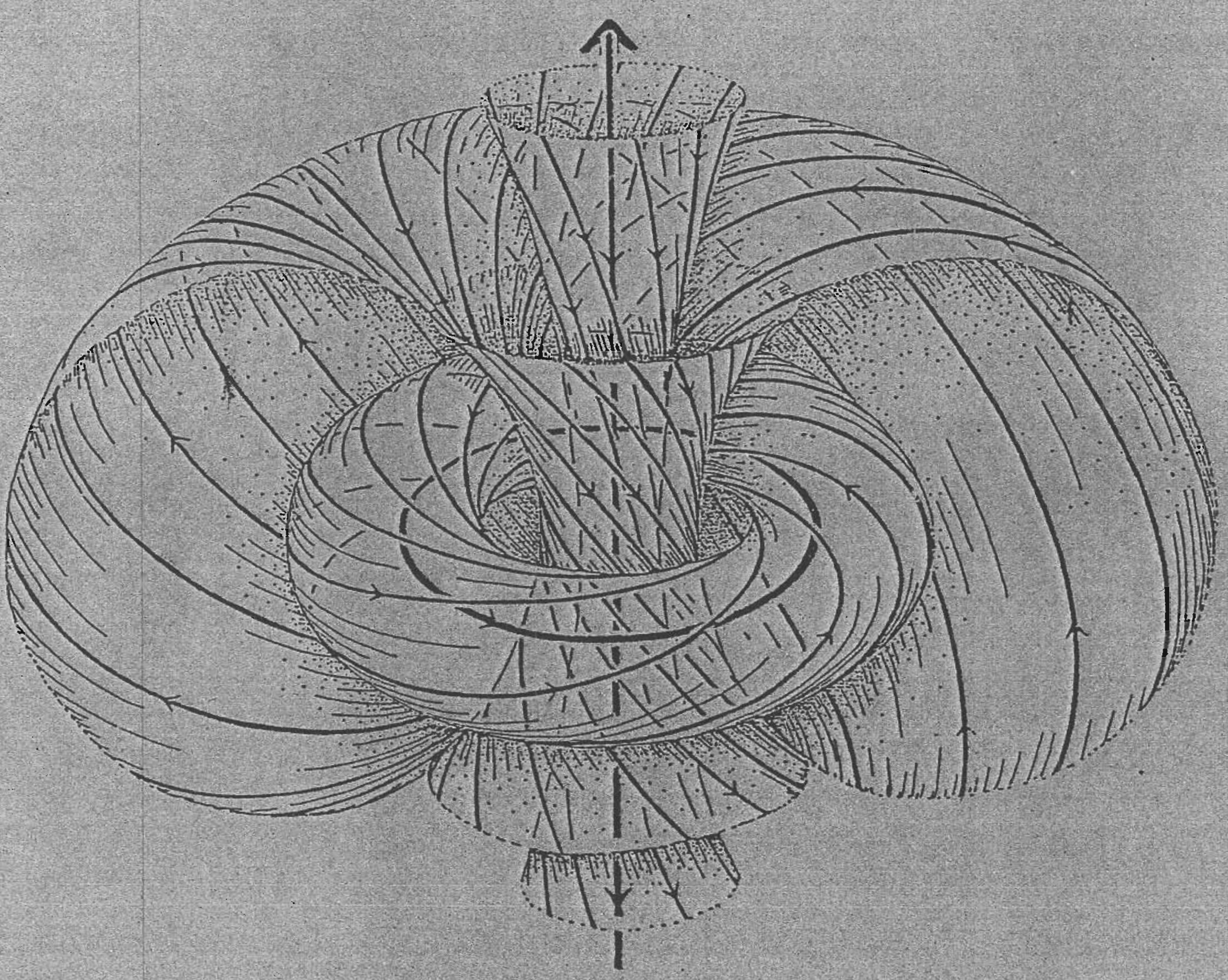
Twistor Newsletter (no 26: 21 March, 1988)



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Twistors and conformal field theory

I shall begin with a brief comparison of twistors in two and four dimensions. In the following table, A and A' are usual 2-spinor indices and we write $\mathbb{P}^A = \mathbb{P}(\mathbb{C}^A)$, $\mathbb{P}^\alpha = \mathbb{P}(\mathbb{T}^\alpha)$, etc.

	2 dimensions	4 dimensions
real space-time	$M_2^\# = S^1 \times S^1$	$M^\# = S^3 \times S^1$
$\mathbb{C}$ space-time	$\mathbb{C}M_2^\# = \mathbb{P}^A \times \mathbb{P}^{A'}$	$\mathbb{C}M^\# =$ quadric in $\mathbb{C}P^5$
twistor spaces	$\mathbb{C}^A, \mathbb{C}^{A'}, \mathbb{C}_A, \mathbb{C}_{A'}$	$\mathbb{T}^\alpha, \mathbb{T}^{\alpha'}, \mathbb{T}_\alpha, \mathbb{T}_{\alpha'}$
"ε-object"	$\varepsilon_{AB}, \varepsilon_{A'B'}$, etc	$\varepsilon_{\alpha\alpha'}$, etc
reality structure	$\lambda^A \mapsto \hat{\lambda}^A, (\hat{\lambda}^0, \hat{\lambda}^1) = (\bar{\lambda}^1, -\bar{\lambda}^0)$	$Z^\alpha \mapsto \bar{Z}^{\alpha'}$
twistor correspondences	$\mathbb{C}M_2^\# = \mathbb{P}^A \times \mathbb{P}^{A'} \to \mathbb{P}^A$ $\mathbb{C}M_2^\# = \mathbb{P}^A \times \mathbb{P}^{A'} \to \mathbb{P}^{A'}$	(well-known)

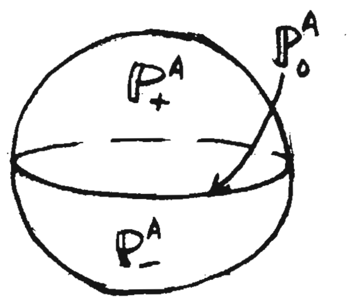
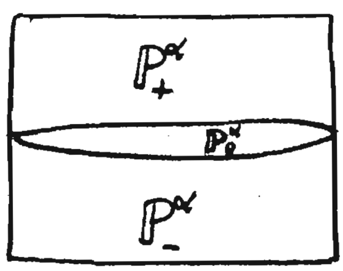
Remarks

1. The conformal structure of $\mathbb{C}M_2^\#$ is supposed given by its product structure, the two families of rulings being the two families of null geodesics.
2. The data in the above table are consistent with the standard way of thinking of twistors as spinors for the conformal group [1]. Care is required in dimension 2, however, since the group of conformal motions of $M_2^\#$ is infinite dimensional. The "correct" conformal group, $O(2,2)$ is characterized as being the group of holomorphic

conformal motions of $\mathbb{C}M_2^{\#}$ which carry the real slice $M_2^{\#}$ to itself.

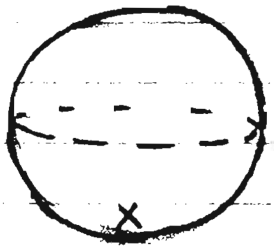
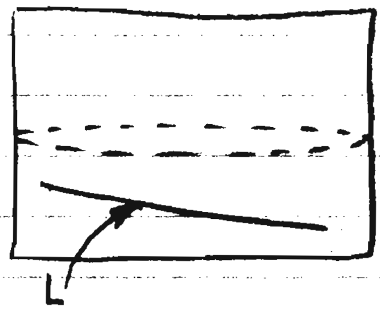
3. Note the differences in the ε-objects and the reality structures [1] between the two cases: and note also that although the twistors for two dimensions look like ordinary 2-component spinors, the conjugation is different (because we're interested in $O(2,2)$ instead of $O(1,3)$). Of course one usually uses $\varepsilon_{\alpha\alpha'}$ to eliminate all primed twistor indices: for example the familiar conjugation $Z^\alpha \mapsto \bar{Z}_\alpha$ is given by $\bar{Z}_\alpha = \bar{Z}^{\alpha'}\varepsilon_{\alpha\alpha'}$.

The basic ingredient of a conformally invariant quantum field theory (CFT) is a Hilbert space H of states. For conformal invariance one tends to take H to be a space of positive-energy solutions W^+ to some massless field equations or a Fock space modelled on W^+. In both dimensions we are considering the twistor construction of W^+ is particularly elegant:

decomposition of projective twistor-space under reality structure		
Typical example of W^+	$\Gamma(\mathbb{P}^A_+, \mathcal{O}(k-1))$	$H^1\left(\mathbb{P}^\alpha_+, \begin{matrix}\mathcal{O}(k-2)\\ \oplus \\ \mathcal{O}(-k-2)\end{matrix}\right)$

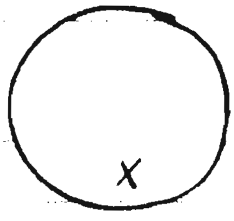
Here I have written $\mathbb{P}^\alpha_+$ for the <u>closure</u> of $\mathbb{PT}^+$ and $\mathbb{P}^\alpha_0$ for $\mathbb{PN}$. Similarly, in the two-dimensional picture I'm thinking of $\mathbb{P}^A_+$ and $\mathbb{P}^A_-$ as the <u>closed</u> hemispheres. Strictly speaking, W^+ for the two-dimensional case should be the holomorphic sections over $\mathbb{P}^A_+$ modulo the global sections.

Thus in each case, the top half of twistor space defines W^+ in a natural way. An alternative way of constructing W^+ is to consider elementary states which give an orthogonal basis for W^+ (see [2] for more details). In each case the elementary states are defined on the punctured twistor space: when I speak of a punctured Riemann surface I shall always mean that finite number of points have been removed, but by "punctured complex 3-manifold" I shall mean that a finite number of (non-intersecting) projective lines have been removed. To get a basis for W^+ the only condition is that the puncture must be in the interior of the bottom half of twistor space.

punctured twistor space		
space of elementary states	$\Gamma(\mathbb{P}^1-\{pt\},\mathcal{O}(k-1))$	$H^1\left(\mathbb{P}^3-L,\ \begin{matrix}\mathcal{O}(k-2)\\ \oplus\\ \mathcal{O}(-k-2)\end{matrix}\right)$

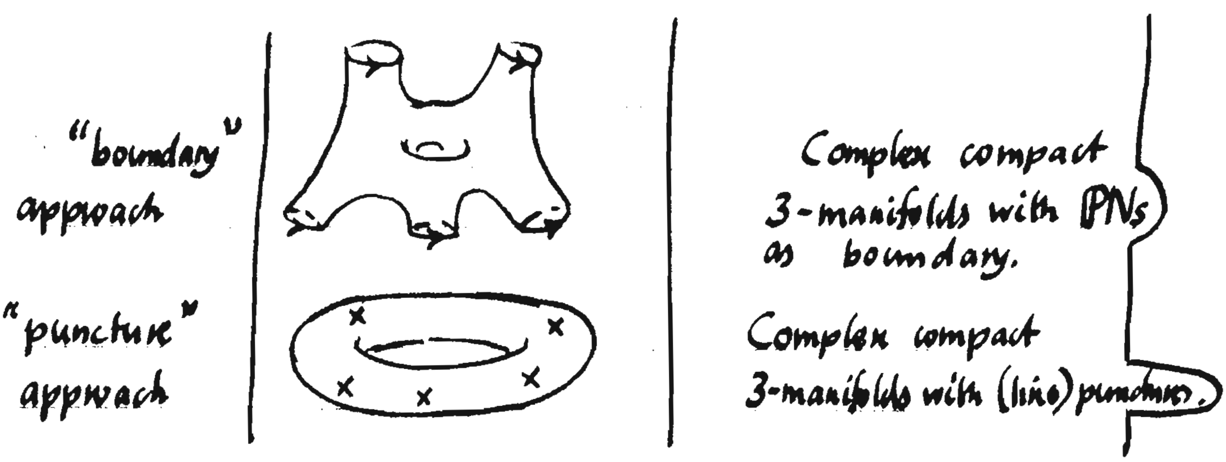
All this is fine for a twistorial free CFT: but what about interactions? In two dimensions there are two (nearly equivalent) ways to proceed [3]. In the present language one replaces the Riemann surface

or



involved in the "free theory" by an arbitrary compact Riemann surface X with (oriented, parametrized) boundary circles or with punctures (with holomorphic coordinates near each puncture). For such surfaces, there are natural ways of glueing them together (for these one needs parametrized boundaries or coordinates near the punctures). For punctures P and Q with z a holomorphic coordinate vanishing at P and w a holomorphic coordinate vanishing at Q, the glueing is defined by the identification $zw = 1$. A reinterpretation in terms of spinor (i.e. two-dimensional twistor) contour integrals can be found in A.P.H.'s article in this TN.

On the other hand, R.P. (this TN) shows how the analogous constructions can be made in four dimensions by glueing bits of twistor space together across $\mathbb{P}N$ boundaries or punctures. It is an interesting feature of the higher-dimensional case that there is <u>less</u> freedom in glueing boundaries together (both assumed to be copies of $\mathbb{P}N$) on account of the rigid CR structure of $\mathbb{P}N$. This is in contrast to the infinite-dimensional Diff(S^1) freedom which appears in two dimensions. Thus by glueing pieces of twistor spaces together we can construct higher dimensional analogues of Riemann surfaces together and glue them together.



Although this extension of the analogy is rather satisfactory it is only part of what is required for the construction of an interacting CFT.

Returning to the two-dimensional case, one selects a Hilbert space H and imagines attaching a copy of H to each boundary circle or puncture of the Riemann surface X. (Actually, for what I'm about to say, one attaches $\bar{H} = H^*$ to any negatively oriented boundary circle.) Then to X one assigns an amplitude on the tensor product of the attached Hilbert spaces. This assignment is to satisfy various conditions, the most important being its "naturality" under glueing operations.

How such an assignment of amplitudes could be achieved in the 4-dimensional case is not known at present.

If the "punctures" approach were adopted, one way to proceed would be to assign an amplitude to $\mathbb{P}^\alpha - \{3 \text{ lines}\}$ and to give a rule for the effect of glueings on the amplitudes. In that way, one could assign an amplitude to all of R.P.'s glued-together twistor spaces. One would expect vertex operators (see T.S.T. in this TN) to play an important part in such an approach.

In the two dimensional case, a popular choice for H is a Fock space on some W.$^+$ Then in the "boundary" picture the amplitudes can nearly be constructed from the subspace structure

$$\mathcal{O}(X) \subset C^\infty(\partial X)$$

induced by restriction of holomorphic functions on X to the boundary. (See G.B.S. – in person – for more details.) The "physical interpretation" of such a theory is then in terms of scattering of strings, the different Fock-space sectors being reinterpreted (roughly speaking) as giving the different modes of the string. It seems quite likely that such a Fock-space theory could be built in four dimensions too, but one would then want a non-stringy reinterpretation of the Fock-space sectors. Is it possible that these could represent different particles, along the lines of the twistor particle programme?

If on the other hand one could construct a two-dimensional theory with $H = W^+$, that might give some clues for a four-dimensional theory with $H = W^+$.

Any possible link with twistor diagram theory is obscure to the author (but see A.P.H. in this TN). Notable by its absence from the above is dual twistor space. It may be, however, that the proposed extension to four dimensions should be modified to allow gluings of twistor space to dual twistor space, even though the two-dimensional theory is formulated solely in terms of one twistor space. One reason for believing such a thing is the differences in the actions of the reality structures alluded to under (3) above. At that level, the two 2-dimensional twistor spaces are, after all, completely unrelated to eachother, whereas the twistor spaces are linked by the conjugation in four dimensions.

I acknowledge many useful talks with R.J.B., E.D., A.P.H., R.P., G.B.S., T.S.T.

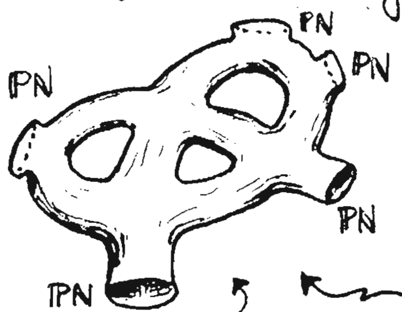
M.A. Singer

References

[1] R.P. and W.R. Spinors and space-time vol. 2, Appendix.
[2] M.G.E. and A.M.P. On the density of elementary states, TN 16.
[3] For "boundary" approach: G.B. Segal: The definition of conformal field theory, to appear.
For "punctures" approach: C. Vafa: Conformal theories and punctured surfaces, preprint.

Pretzel Twistor Spaces

In an accompanying article in this TN, M.A.S. has provided



a very appealing suggestion for combining some of the attractive features of string theory / conformal field theory with twistor theory. As an analogue of the (bounded) Riemann surfaces of str./conf. fld. theory (pretzels), one imagines a complex manifold — say of 3 dimensions or 4 dimensions, according to whether we are considering projective or non-projective twistors — with a boundary consisting of a number of copies of $\mathbb{P}\mathbb{N}$ or $\mathbb{N}$, as the case may be. (See also T.S.T.'s article for a suggestion whereby these $\mathbb{P}\mathbb{N}$s shrink down to lines.) The most appropriate formulation of this idea remains somewhat unclear as of now, but we might think in terms of some sort of analogue of the Fock spaces that in str./conf. fld. th. are to be specified at each disconnected portion of the boundary — say, at each "end". Then amplitudes could be obtained when elements of the appropriate Fock spaces are specified at each end.

From the point of view of twistor theory, a Fock space (in its normal physical interpretation) might seem inappropriate. For reasons of wishing to tie up this idea with twistor diagram theory — ideas due to A.P.H. (see this TN) — and of exploiting the relevant crossing symmetry, spin-statistics, duality, etc., we might prefer to think of the "ends" as corresponding to individual particles, rather than states involving an indefinite number of particles as described by Fock space elements.

The twistorial description of Fock space elements would be by certain elements of

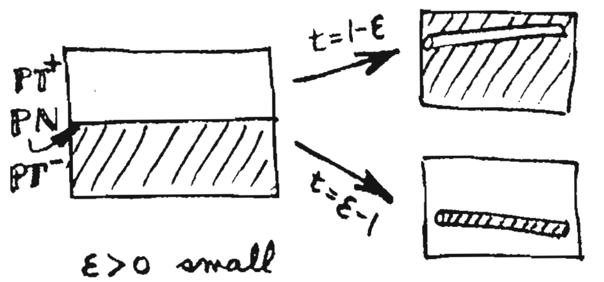
$$\mathbb{C} \oplus H^1(\mathbb{T}^+, \mathcal{O}) \oplus H^2(\mathbb{T}^+ \times \mathbb{T}^+, \mathcal{O}) \oplus H^3(\mathbb{T}^+ \times \mathbb{T}^+ \times \mathbb{T}^+, \mathcal{O}) \oplus \dots$$

i.e. by collections of "twistor functions"

$$f_0,\ f_1(Z^\alpha),\ f_2(Z^\alpha, Y^\alpha),\ f_3(Z^\alpha, Y^\alpha, X^\alpha),\ \dots$$

which are, respectively "0-functions", 1-functions, 2-functions, 3-functions, etc., where the n-function f_n is the twistor wave function for a state consisting of n massless particles. (The "0-function" f_0 would simply be an element of $\mathbb{C}$, defining the amplitude for zero particles.) Instead of this, we might well prefer to think of $f_0, f_1, f_2, f_3, \ldots$ as referring to the functions of several twistors which come up in the <u>twistor</u> <u>particle</u> <u>programme</u>. Thus, to a "first approximation", f_1 might describe the amplitude for the state of a massless particle, f_2 the state of a lepton (?), f_3 the state of an "ordinary" hadron (??), f_4, etc., for more "exotic" hadrons (???) (and f_0 for a "nothing" ..?...?), but an actual massive particle might tend to have small contributions all along the line. We expect f_1 to be a 1-function and suspect that f_2 might perhaps be a <u>relative</u> 1-function (or 2-function?) relative to some sort of diagonal locus in $\mathbb{T} \otimes \mathbb{T}$, conformal invariance being broken at this stage (so mass and $I_{\alpha\beta}$ come in). Perhaps <u>all</u> the f_i are relative 1-functions (or 2-functions) or something (whence $f_0 \equiv 0$). Perhaps, as with strings, the higher fs describe higher "modes" — oscillation of strings being replaced by deformations of $\mathbb{P}\mathbb{N}$?? Perhaps, in accordance with an idea floated by G.B.S., they refer to different jet bundles over $(\mathbb{P})\mathbb{N}$??

All this is extremely vague and speculative, as yet. Let us backtrack a little and try to see whether there are, indeed, any interesting "pretzel" twistor spaces. As with str./conf. theories, <u>closed</u> pretzel spaces are of particular interest. The simplest way to construct such a twistor space (considering the case of pretzelized $\mathbb{P}\mathbb{T}$s only) would seem to be the following. Consider first the two limiting projective motions of $\mathbb{P}\mathbb{T}$:



$\varepsilon>0$ small

These may be achieved by

$$Z^\alpha \leadsto \begin{bmatrix}\omega^0\\ \omega^1\\ \pi_{0'}\\ \pi_{1'}\end{bmatrix} \longmapsto \begin{bmatrix}1&0&t&0\\0&1&0&t\\t&0&1&0\\0&t&0&1\end{bmatrix}\begin{bmatrix}\omega^0\\ \omega^1\\ \pi_{0'}\\ \pi_{1'}\end{bmatrix} = \begin{bmatrix}\omega^0_t\\ \omega^1_t\\ \pi_{0't}\\ \pi_{1't}\end{bmatrix} \leadsto Z^\alpha_t$$

with $t \in (-1,1)$, the two limiting cases being $t \to 1$, $t \to -1$, respectively.

The 5-surface into which $\mathbb{P}N$ is carried (i.e. given by $Z^\alpha_{-t}\bar{Z}_{-t\,\alpha} = 0$) is another copy of $\mathbb{P}N$ (identical with it as a CR-manifold) — call it $\mathbb{P}N_t$. As $t \to 1$, this 5-surf. closes in on the line $L^+ \subset \mathbb{P}\mathbb{T}^+$ given by $Z^\alpha = \begin{pmatrix}\lambda\\ \mu\\ \lambda\\ \mu\end{pmatrix}$, and as $t \to -1$ it closes in on $L^- \subset \mathbb{P}\mathbb{T}^-$, given by $\begin{pmatrix}\lambda\\ \mu\\ -\lambda\\ -\mu\end{pmatrix}$.

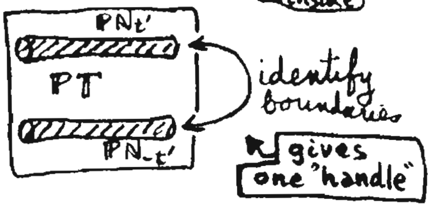
Note that if we define $|Z|^2 = |\omega^0|^2 + |\omega^1|^2 + |\pi_{0'}|^2 + |\pi_{1'}|^2$, then $|Z_t|^2 = (1+t^2)|Z|^2 + 2t Z^\alpha \bar{Z}_\alpha$ and

$Z_t^\alpha \bar{Z}_{t\,\alpha} = (1+t^2) Z^\alpha \bar{Z}_\alpha + 2t|Z|^2$. Hence $\mathbb{P}N_t$ is defined by

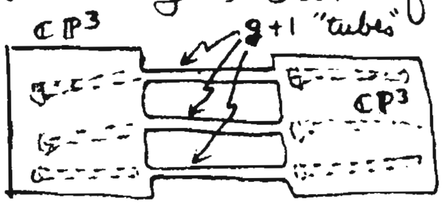
$$Z^\alpha \bar{Z}_\alpha : |Z|^2 = 2t : 1+t^2.$$

$\mathbb{P}N_{1-\varepsilon}$ (resp. $\mathbb{P}N_{\varepsilon-1}$) is the boundary of a tubular neighbourhood of L^+ (resp. L^-).

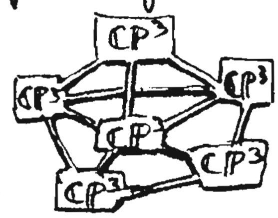
The simplest pretzel space $\mathcal{P}_1$ is given by identifying Z^α with Z_t^α for some fixed t in $(0,1)$ (and hence, up to proportionality, Z^α is also identified with $Z^\alpha_{\tanh(n\tau)}$ where $t = \tanh\tau$, for all $n \in \mathbb{Z}$). This can also be achieved by drilling out the regions above (inside) $\mathbb{P}N_{t'}$ and below (inside) $\mathbb{P}N_{-t'}$ (where $t' = \tanh\frac{1}{2}\tau$) and identifying the boundaries. There is actually some freedom in how such an identification is made, and this allows different pretzel twistor spaces to be built in this way.



The spaces $\mathcal{P}_1$ are the analogues of toruses, i.e. of Riemann surfaces of genus 1. For higher genus (genus g), we can carry out several such identifications simultaneously to give a "$\mathbb{CP}^3$ with g handles". Perhaps easier to visualize, but equivalent, is to take two copies of $\mathbb{CP}^3$ and to identify across from one to the other.



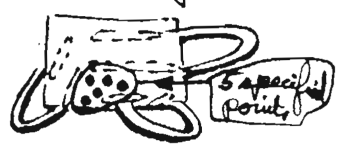
We could also do



, etc., but this gives us no more generality. (To see this, clip all the tubes necessary to make a tree; then fit all the $\mathbb{CP}^3$s inside one of them and we are back with the case of "handles" as before, when we reglue where we had clipped.)

The analogies between these spaces and Riemann

<u>surfaces</u> are quite striking. In particular, we can ask for the <u>dimension</u> m of the <u>moduli</u> <u>space</u> for pretzel twistor spaces of genus p (i.e., how many complex parameters are needed to characterize a $\mathcal{P}_g$, where a $\mathcal{P}_g$ is a $\mathbb{CP}^3$ with g "handles" of this type). (I don't know whether this "genus" corresponds to something standard in algebraic geometry.) Recall that for Riemann surfaces the answer (due to Riemann) is $3g-3$, except when $g=0$ or 1 — where for $g=0$ the answer is 0 and for $g=1$ it is 1. For $\mathcal{P}_g$ the answer turns out to be precisely <u>5 times</u> as large — with the <u>single</u> <u>exception</u> that for $\mathcal{P}_1$, $m=3$ (instead of 5). The proof is similar to that for Riemann surfaces. Think first of a <u>labelled</u> $\mathbb{CP}^3$. The labelling can be achieved by specifying 5 points in general position on $\mathbb{CP}^3$. There are 15 complex degrees of freedom in the specification of each pipe.

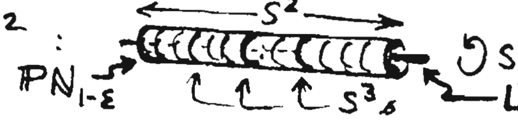


(Or do it with

, which may be a little easier to visualize.) The 15 comes from the size of the projective group on $\mathbb{CP}^3$ $(15 = 4^2-1)$. We have $15g$ for the number of parameters needed to define a <u>labelled</u> $\mathcal{P}_g$. We have to factor out by the freedom in doing the labelling — which is 15 parameter's worth <u>unless</u>, in the <u>generic</u> case, there are continuous (holomorphic) symmetries of $\mathcal{P}_g$. We do <u>not</u> factor out by the motions of the labelling corresponding to a symmetry. Suppose that there are d dimensions of symmetries. Then we get

$$m = 15g - (15-d).$$

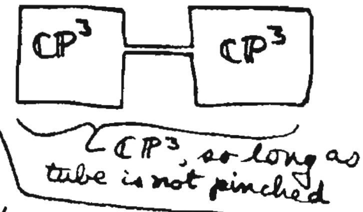
When $g=0$ then clearly $d=15$, so $m=0$. When $g=1$ then a <u>direct</u> argument shows that $m=3$ (whence $d=3$ in the generic case). (This argument is: $\mathcal{P}_1$ is defined by $Z^\alpha \equiv T^\alpha{}_\beta Z^\beta$, up to proportionality, for some fixed $T^\alpha{}_\beta$. The ratios of the eigenvalues of $T^\alpha{}_\beta$ give the moduli of $\mathcal{P}_1$.) It is not hard to see that $d=0$ whenever $g \geq 1$ (as with Riemann surfaces), so $m = 15g-15$ in these cases.

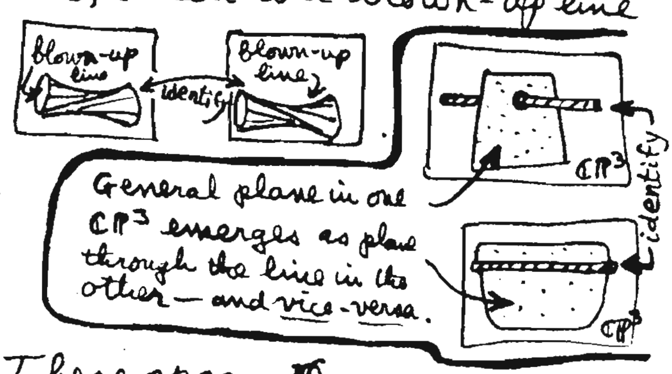
I should remark that there is a subtlety involved in the gluing of the pipes together ("handles"). When we think of $\mathbb{P}N_{1-\varepsilon}$ surrounding L^+, we think of it as an $S^2 \times S^3$ for which the S^3s surround the various points of L^+ (in directions normal to L^+) and where the different points of L^+ give the S^2: 

. If we glue this to $\mathbb{P}N_{\varepsilon-1}$ we must do so in such a way that the small S^3s of $\mathbb{P}N_{1-\varepsilon}$ are stretched the length of $\mathbb{P}N_{\varepsilon-1}$, 

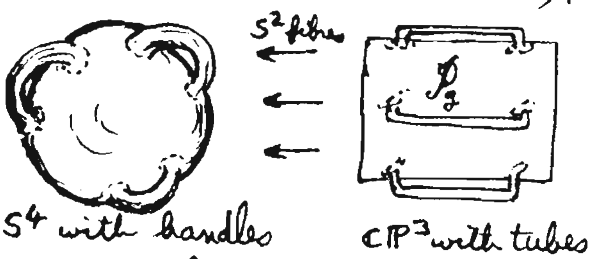
and vice versa. This is possible because of a topological relation which I write symbolically as $S^2 \times S^3 \cong S^3 \times S^2$, each side being a circle bundle over $S^2 \times S^2$, but where the twist can be transferred from being over the second S^2 factor to over the first. (This is the Clifford-Hopf bundle of S^1 over S^2 to give S^3.) If we take planes through L^-, they sweep out one family of S^3s on each $\mathbb{P}N_t$ ("small S^3s" when $t = 1-\varepsilon$, and running the "length" of $\mathbb{P}N_t$ when $t = \varepsilon - 1$); planes through L^+ sweep out the other family of S^3s ("long" when $t = 1-\varepsilon$ and "small" when $t = \varepsilon - 1$). The relation "$S^2 \times S^3 \cong S^3 \times S^2$" can also be seen by examining pairs of orthogonal unit vectors at the origin of $\mathbb{R}^4$. Fix attention on one vector: it sweeps out an S^3 while the other gives a (trivial) bundle over it (trivial since S^3 is parallelizable). Then think of the vectors in the other order.

If we consider a thin tube, where the tube narrows down to nothing, we get in the limit the space (considered by twistorians in connection with null lines on $\mathbb{C}\mathscr{P}$, and by S.K.D. in a context similar to the present one) which is two $\mathbb{C}\mathbb{P}^3$s joined along a quadric, which is a blown-up line in each, and identified with generator systems reversed:





The topology of $\mathscr{P}_g$ is an S^2 bundle over S^4 with g handles ($S^3 \times \mathbb{R}$ handles).



← These spaces $\mathscr{P}_g$ are ones where there are $\mathbb{C}\mathbb{P}^1$s with neighbourhoods which are identical with portions of $\mathbb{C}\mathbb{P}^3$ — and are twistor spaces of conformally flat 4-spaces. I think they are the only ones.

Much more can be said. Of course, in line with non-linear graviton constructions we shall want to deform these spaces; also to deform the CR-structure of $\mathbb{P}N$, etc. Work in progress.

Thanks to M.A.S., A.P.H., G.B.S., T.S.T., E.D., R.B., S.K.D.

~ Roger Penrose

A possible role of vertex operators in Singer's picture of 4-dim CFT

Singer proposes (see this issue of TN) that one can view the Riemann surfaces of 2-dim. conformal field theory as a 2-dim. analogue of twistor space. This leads to the further proposal that 4-dim. CFT can be obtained through replacing the Riemann surface (with boundary a collection of S^1) by a complex 3-dim. manifold (with boundary a collection of $\mathbb{P}N$).

Now one way to make contact with the physical world of interactions in the 2-dim. theory is to introduce vertex operators. For simplicity, consider a cylinder with two bounding circles. If we think of these two circles as representing an incoming and an outgoing state (just as in string theory), then by conformal invariance the bounding circles can be shrunk to points (not entirely clear how, but universally accepted by physicists) and the Riemann surface becomes a sphere with two punctures:

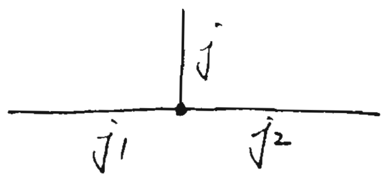
The "physics" is then represented by local operators called vertex operators inserted at these points. These vertex operators keep track of the momenta, positions and quantum numbers of the particles states. For spinless particles one can take e.g. $V_0(k,z) = e^{ik\cdot X}$, and for spin 2, $V_2(k,z) = \partial_a X^b \partial^a X^c e^{ik\cdot X}$, where k = momentum and $X = X(z)$ = coordinate. To get the N-point amplitude one then takes the vaccuum expectation value of the product of N such vertex operators:

$$A = \langle \prod_{i=1}^{N} V_{J_i}(k_i, z_i) \rangle .$$

E.g. $N=4$ leads to the well-known Veneziano amplitude involving hypergeometric functions. Depending on the particular problem, these functions of k and z can be integrated w.r.t. either variable, and the resultant functions are also called vertex operators.

The Twistor picture is tantalizingly similar. If we take a line in $\mathbb{PT}$ we can choose a tubular neighbourhood of it to make it look like a standard $\mathbb{PN}$. (See RP's article in this issue). Since these $\mathbb{PN}$ are to play the role of the S^1 in the 2-dim. theory, the lines whose neighbourhoods they are then correspond to the points (or vertices) at which one can insert vertex operators. But a line in $\mathbb{T}$ corresponds to a point in 4-dim. "space-time", and since such a line is in a $\mathbb{PN}$ the point is "real". So this looks the right object to which one can attach vertex operators. The obvious suggestion is to consider elementary states. Unfortunately I do not know how to do this concretely at present. Perhaps one can think of a different kind of "Penrose transform".

A different way to look at vertex operators is in the representation theory of Diff S^1. They correlate different Verma modules rather like the way Clebsch-Gordon coefficients connect different spins. In fact, I think that it is "pictorially" correct to say that vertex operators are souped-up continuum versions of Clebsch-Gordon coefficients which are just numbers. Roughly, let l be a non-negative integer and j a half-integer s.t. $0 \leq 2j \leq l$. Consider the affine Lie algebra of $sl(2,\mathbb{C})$, and denote by V_j the subspace of the integrable highest weight module corresponding to j, determined by a certain vacuum condition. These V_j are irreducible $sl(2,\mathbb{C})$-modules of dimension $2j+1$. Then the Virasoro algebra (centrally extended Lie algebra of Diff S^1) acts on each of these V_j. Given a vertex



satisfying $|j_1 - j_2| \leq j \leq j_1 + j_2$, $j_1 + j_2 + j \in \mathbb{Z}$, then $\exists$ a unique vertex operator for each j, mapping

$$V_j \otimes V_{j_1} \longrightarrow V_{j_2}.$$

Now inside Diff S^1 there is the important subgroup $SL(2,\mathbb{C})$ or $SU(1,1)$. While in 4 dimensions we do not have a corresponding infinite-dim. conformal group, twistor theory has $SL(4,\mathbb{C})$ or $SU(2,2)$. One can hope to turn the tables round and use some of the recent results on representations (e.g. the "discrete series") in twistor theory to get a handle on the proper twistorial form of vertex operators for a 4-dim. theory. The close relation between the two representation theories was expounded by RJB in a recent QFT seminar.

Thanks to RJB, RP and MAS. Tsou Sheung Tsun.

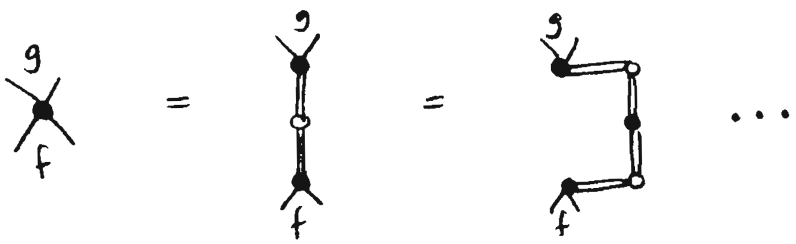
Conformal Field Theories and Twistor Diagrams

In an earlier article (**TN 23**) I emphasised the vital importance of locating a prescriptive theory of fundamental physics of which twistor diagrams could be the evaluative calculus (in analogy to Feynman diagrams). I commented on the appearance of the vertices

in twistor diagrams for massless electroweak theory, hazarding the suggestion that such diagrams might be generated systematically by a combinatorial rule based on such vertices. Such a rule, if it existed, should then be derived from a deeper theory in analogy to the derivation of the Feynman rules from an interaction Lagrangian.

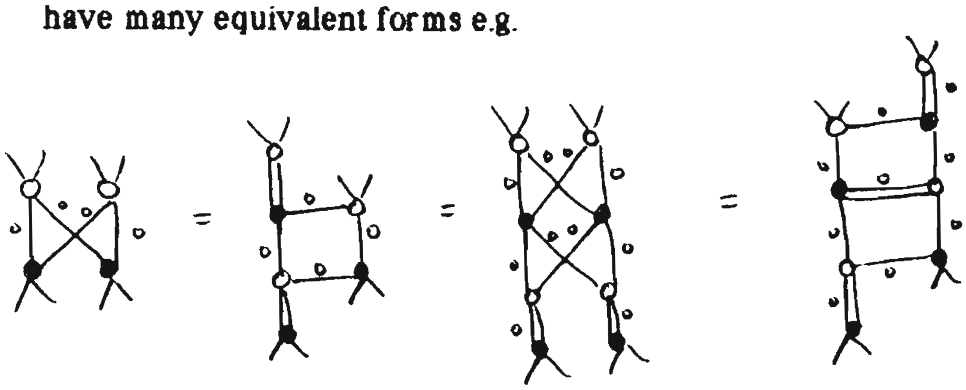
Despite the suggestive features of these twistor diagrams, however, it was not possible actually to establish any such combinatorial rule. There is, furthermore, a prominent feature of twistor diagrams distinguishing them from Feynman diagrams, namely that for any particular amplitude there are many twistor diagram representations. This suggests that the analogy with Feynman diagrams may be indirect.

As examples: even for the zero-order interaction we have



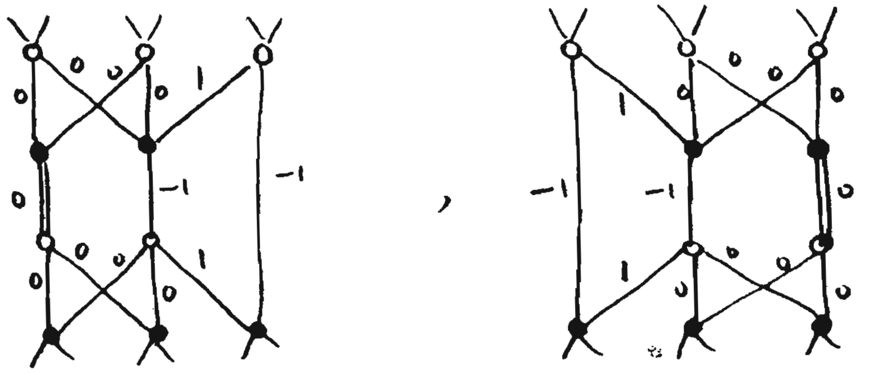
(A)

which is enough in itself to suggest that the "order" of a diagram cannot be defined in terms of the number of its vertices. At the first order level we have many equivalent forms e.g.



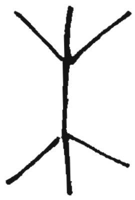
etc. etc.

Likewise, if we consider the higher-order diagrams described in **TN 25** ; we note the equivalence of



etc. (B)

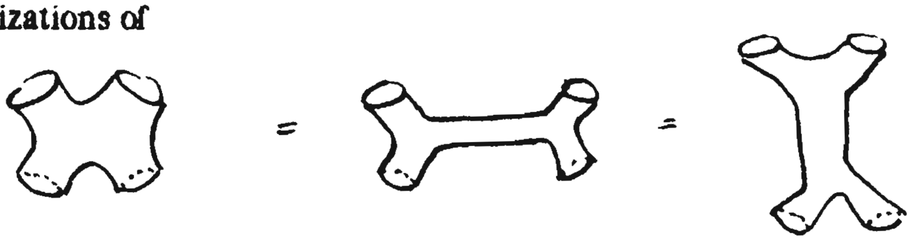
All of these correspond to the Feynman diagram for second-order ϕ^4 scattering



but none of them exhibit the actual symmetry of the amplitude. Reference to that article will show many other examples.

Now R.P. did in fact suggest long ago that there was some similarity to be seen between twistor diagrams and the planar diagrams of the Veneziano dual model, originally devised in the context of describing the strong interaction. As is well known, the identity

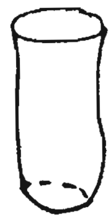
of the planar diagrams can be interpreted in terms of string interactions: both are realizations of



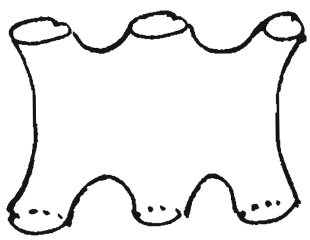
We can therefore ask the question: is there some analogous structure in *twistor* geometry such that the many different equivalent twistor diagrams can be interpreted as different ways of evaluating an amplitude properly defined on that structure? This question could have been asked at any time in the last 15 years or so, and it is hard to see why we have not addressed it before. However, our recent exposure to conformal field theories, with its emphasis on *complex manifold* structure, has not only prompted the question more acutely than before but has stimulated a specific suggestion for what this structure could be (see Mike Singer, Florence Tsou, Roger Penrose, this **TN**): namely (i) the interpolation of complex manifolds between copies of **PN** and (ii) in some way specifying free in- and out-fields on those copies of **PN**, (iii) in some way analytically continuing such data across the interpolating manifolds and then combining them to give a natural functional of the in- and out-states.

Let us adopt M.A.S.'s pictures for this structure. We shall adopt the interpretation in which the boundaries of the picture are associated with *one-particle* states. In the first instance these are *massless* fields, so that an appropriate H^1 in one twistor variable is prescribed on each **PN** boundary piece. [However, there is room in this scheme, following R.P.'s suggestion, for a two-twistor or n-twistor object to be prescribed on a boundary. This idea opens up a new view of how the twistor representation of a massive one-particle state by n twistors can differ essentially from a massless n-particle state - a question hitherto puzzling fro the point of view of twistor diagram theory.]

We are thus led to hazard the suggestion that all the inner product diagrams (A) might be seen as different evaluations of something like



and the diagrams (B) as evaluations of something of form



(an object in which the true symmetry would be manifest, even though that symmetry is broken when choosing a specific evaluation *via* a twistor diagram.)

If this were so then we would replace the idea of a sum over graphs defined by vertices by a sum over all interpolating complex manifolds. *This* would become the analogy to the summing over Feynman diagrams, and we should then go on to seek some fundamental theory explaining *this* generating rule.

As yet we have no theory that yields a correspondence between the Singer pictures and twistor diagrams. But there are general reasons why one might be hopeful:

(1) Note that (at least in the first instance) we are looking for a twistor-based theory which gives a new description of an essentially well-known flat-space theory of massless fields. We are translating interactions which are described in space-time as interactions at a point. But points are *extended* objects in $\mathbf{T}$. So we should always have *expected* something "stringlike" in $\mathbf{T}$ to emerge.

(2) Again, note that (at least in the first instance, and modulo divergence problems), we know the functionals of free fields that we are looking for - holomorphic conformal invariant linear functionals with various symmetries. If we can find ***any*** way of deriving functionals with these features from a theory based on Singer pictures, then there seems an excellent chance that they will be the right ones.

(3) In looking for a correspondence between Singer pictures and twistor diagrams, we might look first at the very simplest case - the inner product diagrams (A). For a further simplification we might further look at the analogous spinor integrals. Of these, the very simplest example is

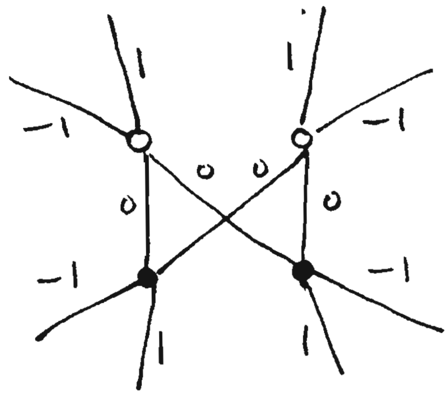
$$\oint f(\lambda)g(\lambda)\,D\lambda = \oint f(\lambda)(\lambda.\mu)^{-1}g(\mu)\,D\lambda\mu = \oint f(\lambda)(\lambda.\mu)^{-1}(\mu.\nu)^{-1}g(\nu)\,D\lambda\mu\nu$$

etc. etc.

These are large-dimensional contour integrals in various products of P^1s But they could be *re-interpreted* as specifying the glueing together of various pieces of P^1s by making the identifications $\lambda=\mu$, $\mu=\nu$, etc., so that each integrals is really being done on the same P^1 manifold, described in different ways.

Although this is a hopeful line of thought, I must say that at present I have no idea how it can be generalised to other homogeneities, or to twistor space in a way that naturally brings in the dual spaces.

Lastly, I refer to my third article in **TN 25**. There it was argued that the twistor diagrams that traditionally have been considered, and such as have been written down above, are not the fundamental objects. They should be thought of as *periods* of the more fundamental but as yet not very well defined integrals given by (e.g.)



These are the objects which are glued together to make twistor diagrams for higher-order amplitudes, i.e. correspond to the combination of the *off-shell* Feynman propagators in Feynman diagrams. One takes various possible periods of these integrals to obtain the amplitudes that arise when the external legs are specified to correspond to *free* in- or out-fields in the various possible channels. Thus I suggest that *these* are the objects that should correspond to the pieces of manifold that are in some sense glued together to build up higher-order Singer pictures. It seems to me therefore that a Singer picture should turn out to specify not an amplitude, but some functional (perhaps not very well defined) whose various *periods* would give the amplitudes in the various different possible channels. Note that inhomogeneity (the "k") and logarithmic propagators were essential in defining these "off-shell" diagrams. I suggest that corresponding [non-obvious] structures would have to be appear in any theory of manifolds which makes sense of the Singer pictures.

Thanks to Mike Singer, Roger Penrose and Florence Tsou -

Andrew Hodges

Geometry on $\mathbb{CP}^1$ and the Virosoro Algebra

It is currently the rage to study a central extension of the Lie algebra of vector fields on the circle, the Virosoro algebra. The purpose of this note is to point out how the cocycle of this extension arises naturally from the geometry of $\mathbb{CP}_1$ under the action of $SU(1,1)$.

The starting point is $\mathbb{CP}_1$ with $\mathcal{U}^\pm = \{z \mid |z| \gtrless 1\}$, $S^1 = \{z \mid |z| = 1\} = N$ with $\Theta = \mathcal{O}(2)$ = sheaf of holomorphic vector fields and the complexification of the Lie algebra of analytic vector fields on the circle

$$V = \Gamma(N, \Theta) = \mathbb{C}\,\mathrm{Vect}(S^1)$$

(meaning sections near N). Mayer-Viëtoris gives $V = V^+ \oplus \mathfrak{sl}(2,\mathbb{C}) \oplus V^-$ where $V^\pm = \Gamma(\bar{\mathcal{U}}^\pm, \Theta)/\mathfrak{sl}(2,\mathbb{C})$ are in the (anti)-holomorphic discrete series for $SU(1,1)$ (cf decompositions of E^8 & F^4 into reps of $SO(16)$ & $SO(8)$, respectively). There is an exact sequence (Bernstein-Gelfand-Gelfand)

$$0 \longrightarrow \mathfrak{sl}(2,\mathbb{C}) \longrightarrow \Theta \xrightarrow{\ \partial^3\ } \Omega^{\otimes 2} \longrightarrow 0 \qquad (1)$$

↑ sheaf of quadratic differentials

characterized by the fact that ∂^3 is left invariant by $SL(2,\mathbb{C})$. The corresponding long exact sequence gives

$$0 \longrightarrow \mathfrak{sl}(2,\mathbb{C}) \longrightarrow V \xrightarrow{\ \partial^3\ } \Gamma(N, \Omega^{\otimes 2}) \longrightarrow \mathfrak{sl}(2,\mathbb{C}) \longrightarrow 0$$

Put $L_k = z^{-k+1}\frac{d}{dz}$ so $[L_k, L_\ell] = (k-\ell)L_{k+\ell}$ and $\omega_k = z^{k-2}(dz)^2$ so $\partial^3 L_k = k(1-k^2)\omega_{-k}$. There is a natural pairing, for $\alpha \in V$, $\omega \in \Gamma(N, \Omega^{\otimes 2})$ given by

$$\langle \alpha, \omega \rangle = [\delta(\alpha \lrcorner \omega)] \in H^1(\mathbb{P}, \Omega^1) \cong \mathbb{C}$$

(δ is the M-V connecting homomorphism). Ofcourse $\langle \alpha, \omega \rangle = \frac{1}{2\pi i}\int_{S^1} \alpha \lrcorner \omega$. This gives a skew symmetric 2-form on V by

$$\Omega(\alpha, \beta) = \langle \alpha, \partial^3 \beta \rangle \qquad \text{so that} \quad \Omega(L_k, L_n) = k(k^2-1)\delta_{k,-n}$$

trivial on $\mathfrak{sl}(2,\mathbb{C})$. This is (up to scale) the cocycle defining the Virosoro central extention by

$$[\alpha, \beta]_{\text{Virosoro}} = [\alpha, \beta]_V + \frac{1}{24}\Omega(\alpha, \beta)\,\underset{\sim}{c} \qquad \left(\text{physicists choose } \tfrac{1}{24}\right)$$

($\underset{\sim}{c}$ spans the centre). The Jacobi identity for $[\,,\,]_{\text{Virosoro}}$ is equivalent to the cocycle condition on Ω, ie.

$$\Omega([\alpha, \beta], \gamma) + \text{cyclic permutations} = 0$$

which follows because $[\alpha, \beta]\,\partial^3 \gamma$ + cyclic perms $= d([\alpha, \beta]\,\partial^2 \gamma$ + cyclic perms) (in the langauge of Verma modules, there is a homomorphism of $[V(2)^{\otimes 3}]^* \to N(-2))^*$ factoring through $V(0)^*$).

Now it is still rather mysterious why the Virasoro cocycle should arise like this: usually, it comes from an embedding of Vect(S^1) in a loop algebra & the Virasoro algebra is the pull back of the Kacs-Moody central extension of that:

$$\begin{tikzcd} \text{Virasoro} \arrow[r] \arrow[d] & \hat{L}\,sl(2,\mathbb{C}) \arrow[d] \\ \text{Vect}\,S^1 \arrow[r, "\substack{\text{(as vector}\\ \text{fields on}\\ \text{loops)}}"'] & L\,sl(2,\mathbb{C}) \end{tikzcd} \qquad (2)$$

We can hunt around for a geometric explanation of this along the lines of what has just been done. A (complex) loop algebra is given by letting $\mathcal{O}(sl(2,\mathbb{C}))$ = $sl(2,\mathbb{C})$-valued functions on $\mathbb{P}_1$. Naturally, there is a differentially split sequence

$$0 \longrightarrow K \longrightarrow \mathcal{O}(sl(2,\mathbb{C})) \underset{\longleftarrow D}{\longrightarrow} \Theta \longrightarrow 0$$

D fails to be a homomorphism of Lie algebras — but only just: the obstruction is exactly $\alpha\,\delta^3\beta - \beta\,\delta^3\alpha \in \Omega^1$: nonetheless, if we define, for $\xi, \psi \in \Gamma(N, \mathcal{O}(sl(2,\mathbb{C}))) = \mathcal{L}$

$$\Omega_{\mathcal{L}}(\xi, \psi) = [\langle \xi \smile d\psi \rangle_{\text{Killing form}}] \in H^1(\mathbb{P}, \Omega^1) = \mathbb{C}$$

$$= \frac{1}{2\pi i} \int_{S^1} \langle \xi, d\psi \rangle_{\text{Killing form}}$$

then, remarkably, $\Omega_{\mathcal{L}}(D\alpha, D\beta) = \delta\Omega(\alpha, \beta)$ so D would appear to be trying to understand (2). If one understood why this identity was true and its relation to (2) one might be able to construct representations of the Virasoro algebra from the Geometry of the projective line.

<u>Remark</u>: $\Gamma(N, \Omega^{\otimes 2}) \oplus \mathbb{C}$ is a (finite) dual of $\tilde{V}$ = Virasoro algebra. The infinitesimal co-adjoint actions of L_k on $\Omega_{\alpha,\beta,j}$ (where $\Omega_{\alpha,\beta,j} = \beta\omega_j + \alpha\,\underset{\sim}{c}^*$) are given by

$$(\text{coad}\,L_n)\cdot\Omega_{\alpha,\beta,j} = \beta(j-2n)\omega_{j-n} - \alpha(1-n^2)n\,\omega_{-n}$$

Stabilizers of $\Omega_{\alpha,\beta,0}$ are as follows: always L_0 (& if this only, the orbit resulting is Diff S^1/S^1, studied by Rajeev & Bowick); if $2\beta + \alpha(1-n^2) = 0$ then $L_{\pm n}$ stabilize also and the orbit is Diff $S^1 / SL^{(n)}(2,\mathbb{R})$ (an n-fold covering of $SL(2,\mathbb{R}) = SU(1,1)$) (the real structure on V is induced by the conjugate of the derivative of the antipodal map & given by $L_k \longrightarrow L_k^\dagger = -L_{-k}$: real vectors are $i(L_k + L_{-k})$ & $(L_k - L_{-k})$). This latter orbit is not evidently a "complex" manifold unless $n=1$. Observe also that Diff(S^1) has no complexification, for if $\alpha = 0$ then ω_j is stabilized by something only if j is even. On the other hand $\omega_{-1}, \omega_{-2}, \ldots$ lie on the same orbit under any complexification of Diff(S^1) $\longrightarrow$ contradiction

Rob Baston

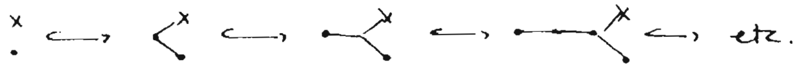
The Geometry of Pure Spinors and Invariant differential operators in higher dimensions.

The Penrose transform for G/P can be used to construct invariant differential operators for forms on $\mathbb{C}S^{2n}$ = $2n$ dim Minkowski space (& hence, via Cartan connections, on all $2n$-dim conformal manifolds). One picks an appropriate homogeneous vector bundle on $Z^{2n} = \bullet\!-\!\bullet\cdots\!<^{\times}$ = projective pure spinors for $SO(2n+2,\mathbb{C})$ (the natural higher dimensional analogue of Twistor space) and calculates its Penrose transform.

The choice of such vector bundles (of zero infinitesmal character) is parameterized by the Hasse diagram of the parabolic $q = \bullet\!-\!\bullet\cdots\!<^{\times}$ (a poset of the Weyl group of $so(2n+2,\mathbb{C})$). This is a directed graph whose nodes correspond to affine cells in Z^{2n} and whose edges give the attaching maps. There are certain specific cells in this specified as the top-dimensional cell in each Z^{2m} in the natural inclusions (given $SO(m,\mathbb{C}) \hookrightarrow SO(m+2,\mathbb{C})$)

$$\begin{array}{ccccccc} Z^2 & \hookrightarrow & Z^4 & \hookrightarrow & Z^6 & \hookrightarrow & \cdots\cdots \\ \| & & \| & & \| & & \\ \mathbb{C}P_1 & & PT & & \text{6 dim'l quadric} & & \end{array} \qquad (1)$$

(Penrose & Rindler, Vol 2 indicates how to build higher spinors from lower ones)
From the point of view of Dynkin diagrams, (1) is just



Let $w_k \in$ Weylgroup of $so(2n+2)$ be the element corresponding to such a cell in Z^{2n-2k}.
The homogeneous bundle on Z^{2n} is $\mathcal{O}_q(w_k \cdot 0)$.

<u>Theorem</u>: $H^*(Z^{2n}, \mathcal{O}_q(w\cdot 0))$ is the total cohomology of

$$E_1^{p,q}: \quad \begin{array}{|cccccccc} & & & \overset{p=n-k+1}{\downarrow} & & & & \\ \Omega^k & \Omega^{k+1} \cdots & \Omega^n_+ & 0 & 0 & \cdots & 0 & \leftarrow q = nk - k(k+1)/2 \\ 0 & 0 \cdots & 0 & 0 & 0 & \cdots & 0 & \\ \vdots & \vdots \cdots & \vdots & \vdots & \vdots & \cdots & \vdots & \\ 0 & 0 \cdots & 0 & \Omega^{2n-k+1} & \Omega^{2n-k+2} & \cdots & \Omega^{2n} & \leftarrow q = n(k-1)-(k-1)k/2 \\ 0 & 0 \cdots & 0 & 0 & 0 & \cdots & 0 & \\ \hline \end{array}$$

<u>Proof</u>: A calculation in the symmetric group on n-letters.

It follows that there are non-zero maps $D: \Omega^{k-1} \longrightarrow \Omega^{2n-k+1}$ on $\mathbb{C}S^{2n}$ which are "non-standard" (ie, like $\Box^2: \mathcal{O} \longrightarrow \Omega^4$ in four dimensions) (One has still to work hard to prove non-zeroness, here).

<u>Query</u>: can one use this to prove one has <u>all</u> invariant operators between forms?

(Thanks to MGE for very many good discussions)

Rob Baston

A Symplectic Penrose Transform?

The Penrose transform on complex homogeneous spaces is well worked out, now. One might ease the requirement of homogeneity by considering symplectic, Käehler manifolds Z with a symplectic G_0-action; G_0 is a compact Lie group. These yield the "twistor space" side of a double fibration as follows: let G be the complexification of G_0 and $X = G/P$, $P \subset G$ parabolic. So X is homogeneous and there is a moment map $\mu_X : X \longrightarrow \mathfrak{g}_0^*$, given a line bundle $\mathcal{L}$ on X. For $x \in X$ let $\hat{x} = iV_x$ where V_x is the vector in $\mathfrak{g}_0$ corresponding to $\mu_X(x)$ under the Killing form. Let $\mu_Z : Z \longrightarrow \mathfrak{g}_0^*$ be a moment map, also, and let

$$f(z) = \langle \hat{x}\cdot\mu_Z(z), \mu_Z(z)\rangle \qquad (\langle\, ,\rangle : \text{Killing form})$$

It turns out that the maximum of f (which is real, by virtue of $\hat{x}$ being hermitean) is achieved on an even dimensional subvariety of Z (which might be called the coherent subvariety* of Z corresponding to x). It appears this subvariety is a complex subvariety of Z. Indeed, if $Z = G/Q$ (eg., $Y = \mathbb{PT}$, $X = \mathbb{M}$) then this subvariety L_x is exactly the corresponding variety. So we have a "double fibration"

$$\begin{tikzcd} & Y \arrow[dl, "\eta"'] \arrow[dr, "\tau"] & \\ Z & & X \end{tikzcd} \qquad \longleftarrow \text{the "correspondence space".}$$

If the symplectic structure on Z is integral Z has a natural complex line bundle $\mathcal{L}_Z$ (it would be $\mathcal{O}(1)$ for $Z = \mathbb{CP}_3$) and one ought to be able to compute the Penrose transform for $\mathcal{L}_Z^{\otimes p}$

(* this terminology because the construction follows that of coherent states in geometric quantization)

Rob Baston

2 dimensional conformal invariants

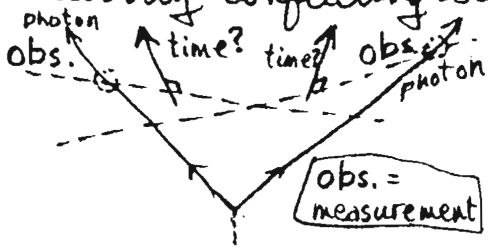
Let Σ be any complex curve with distinguished volume form, thought of as a real manifold. The methods of Ochiai & many others (see my thesis) show that $\exists$ a $\hat{B}$-principal bundle $\hat{P} \longrightarrow \Sigma$ where B is the subgroup of upper triangular matrices in $PSL(2,\mathbb{C})$; If Σ is spin, this extends to $P \longrightarrow \Sigma$, B-principal, B = upper Δ in $SL(2,\mathbb{C})$. The Cartan connections on this structure are not unique (as in higher dimensions than 3) but parameterized by sections Φ of the quadratic forms $\Omega^{\otimes 2}$ of Σ; there ought to be a way of using Verma modules for the Virosoro algebra to construct invariants here. If for some reason Φ is given then the Verma module theory of $sl(2,\mathbb{C})$ will yield differential invariants. If Σ is the lift of a null geodesic in curved M to projectivized spin & spin' bundle $\mathcal{F}_{123}$ then Φ is determined to be $\Phi^{ABA'B'}\pi_{A'}\pi_{B'}\mu_A\mu_B$ and invariants of the conformal structure of M result — see [1], [2]

Rob Baston

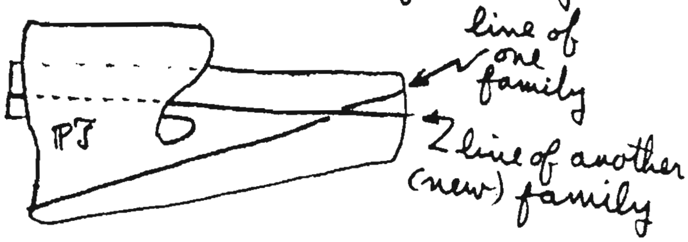
[1] Rod Gover: thesis expected soon.
[2] MGE: TN 20, p40.

Twistors and State-Vector Reduction

One of the most puzzling features of the procedure of the "reduction of the state vector" in quantum theory is that there seems to be no particular "moment" at which it happens. Yet, in the real world of our experience, such discontinuous changes in the quantum-mechanical description of the world have to be considered to take place, generally, between two measurements. One might have thought that any such violently discontinuous change in the state of the world — if it were a real effect — ought to be more noticable, as to *when* it actually takes place! Also, since the Schrödinger equation is such a nice smooth analytic thing, it seems odd that Nature should choose to execute such violent discontinuous jumps from time to time. Perhaps most baffling is the non-local and seeming relativity-conflicting behaviour in E.P.R.-type (Clauser-Aspect) experiments. Spacelike-separated measurements take place. There is a conflict between the apparent time-ordering of the "reductions" due to these two measurements. "*When*" do these reductions "*actually*" take place?



It is conceivable that a twistor-type viewpoint could provide some sort of resolution of this puzzle. Suppose that reduction is a gravitational effect (cf. R.P., TN19) and that the space-time is described twistorially. Of course we need to have solved the problem of representing *general* space-times, not merely anti-self-dual ones, for this to work (beyond the googly!) and it may well be that space-time points are not simply "lines" in twistor space. But for the sake of descriptiveness (only), let's take lines. Now suppose that, with some measurement, the twistor space $\mathbb{PT}$ becomes sufficiently "curved" that the original family of holomorphic lines, representing space-time points, peters out — and we must switch to a *new* family of such lines in order to keep going. Somehow the geometry of space-time seems to jump — yet in the "actuality" of the twistor space there is no jump — just a (necessary) shift in viewpoint. Anyway, it's a thought!



~ Roger Penrose

A TWISTOR TRANSFORM FOR THE DISCRETE SERIES: The case of $SU(1,2)$.

§I. A twistorial alternative to the construction of representations via L^2-cohomology exists for the ladder representations of $SU(p,q)$. These form part of the analytic continuation of the discrete series for those groups. The crucial ingredient of the construction is the twistor transform for cohomology on $\mathbb{CP}^n$:

$$\tilde{S}: H^s(\mathbb{P}^+, \mathcal{X}) \cong H^{s'}(\mathbb{P}^{*-}, \mathcal{X}^\vee)$$

where s and s' are the dimensions of the respective maximal compact subvarieties and $\mathcal{X}$ is a homogeneous line bundle with $\mathcal{X}^\vee = \mathcal{X}^* \otimes \Omega^{top}$. What follows is an example of how to do something similar for the discrete series. The essential isomorphism holds for arbitrary semi-simple Lie groups. We will show how it works for $SU(1,2)$. The case of $SU(1,1)$ is a good exercise.

Let's set the notation:

$\mathbb{F} = \mathbb{F}_{12}(\mathbb{C}^3) = \{(L,P) \mid L = \text{line in } \mathbb{C}^3, P = \text{plane in } \mathbb{C}^3 \text{ and } L \subseteq P\}$.

$\mathbb{P} = \mathbb{CP}^2 = \{\text{lines in } \mathbb{C}^3\}$

$\mathbb{F}_2 = G_2(\mathbb{C}^3) = \{\text{planes in } \mathbb{C}^3\} \cong \mathbb{P}^*$

Φ = Hermitian form of signature $(+--)$ which defines $SU(1,2)$

There is the double fibration:

$$\begin{tikzcd}
 & \mathbb{F} = \times\!-\!\times \arrow[dl] \arrow[dr] & \\
\mathbb{P} = \times\!-\!\bullet & & \mathbb{F}_2 = \bullet\!-\!\times
\end{tikzcd}$$

Setting $G = SU(1,2)$, the G-orbits on these flag varieties are of the form:

$\mathbb{F}^{+,+-} = \{(L,P) \in \mathbb{F} \mid \Phi|_P \text{ has signature } (+-), \Phi|_L \text{ is positive definite}\}$

$\mathbb{P}^- = \{L \in \mathbb{P} \mid \Phi|_L \text{ is negative definite}\}$

$\mathbb{F}^{+-} = \mathbb{F}_2^{+-} = \{P \in \mathbb{F}_2 \mid \Phi_P \text{ has signature } (+-)\}$

and so on. For an explanation of notations involving Dynkin diagrams ($\times\!-\!\bullet$, $\times\!-\!\times$, $\bullet\!-\!\times$, $\bullet\!-\!\bullet$), see the manuscript by MGE & RJB.

§II The Penrose transform produces the isomorphisms:

$$H^1(\mathbb{P}^-, \overset{p}{\times}{-}\overset{q}{\bullet}) \cong H^0(\mathbb{F}_2^{--}, \overset{q}{\bullet}{-}\overset{-p-q-3}{\times}) \qquad p,q \geq 0$$

$$H^1(\mathbb{P}^-, \overset{-p}{\times}{-}\overset{q}{\bullet}) \cong H^0(\mathbb{F}_2^{--}, \overset{q}{\bullet}{-}\overset{p-q-3}{\times}) / \overset{q}{\bullet}{-}\overset{p-q-3}{\bullet} \qquad q \geq 0,\ p \geq q+3$$

$$H^1(\mathbb{F}^{+-}, \overset{p}{\bullet}{-}\overset{q}{\times}) \cong H^0(\mathbb{P}^+, \overset{-p-q-3}{\times}{-}\overset{p}{\bullet}) \qquad p,q \geq 0$$

$$H^1(\mathbb{F}^{+-}, \overset{p}{\bullet}{-}\overset{-q}{\times}) \cong H^0(\mathbb{P}^+, \overset{q-p-3}{\times}{-}\overset{p}{\bullet}) / \overset{q-p-3}{\bullet}{-}\overset{p}{\bullet} \qquad p \geq 0,\ q \geq p+3$$

Here $\overset{a}{\bullet}{-}\overset{b}{\bullet}$ is a finite dimensional representation of $SL(3,\mathbb{C})$, which we will abbreviate to F. In some sense, factoring out by F is like factoring out the constants in other settings. There are the isomorphisms of the varieties:

$$\mathbb{P}^+ \cong \mathbb{F}_2^{*--}$$
$$\mathbb{P}^- \cong \mathbb{F}_2^{*+-}$$

effected by the map $z \longmapsto z^\perp = \{\xi \in (\mathbb{C}^3)^* \mid \xi|_z = 0\}$.

From these, we get the twistor transform, Mk. I:

(A) $$H^1(\mathbb{P}^-, \overset{-p}{\times}{-}\overset{q}{\bullet}) \cong H^0(\mathbb{F}_2^{--}, \overset{q}{\bullet}{-}\overset{p-q-3}{\times}) / F$$

$$\xrightarrow{\tilde{S}} \cong H^0(\mathbb{P}^{*+}, \overset{q}{\bullet}{-}\overset{p-q-3}{\times}) / F$$

The last space is conjugate isomorphic to:

$$H^0(\mathbb{P}^+, \overset{p-q-3}{\bullet}{-}\overset{q}{\times}) / F.$$

From now on, we will ignore the quotient by the finite dimensional subspace. Then from $\varphi, \psi \in H^1(\mathbb{P}^-, \overset{-p}{\times}{-}\overset{q}{\bullet})$ form the cup product

$$\varphi \cup \overline{\tilde{S}\psi} \in H^1(\mathbb{P}^0, \overset{-q-3}{\times}{-}\overset{2q}{\bullet}).$$

Follow this by the Mayer-Vietoris connecting map

$$\delta(\varphi \cup \overline{\tilde{S}\psi}) \in H^2(\mathbb{P}, \overset{-q-3}{\times}{-}\overset{2q}{\bullet}).$$

When $q = 0$, the vector bundle $\overset{-q-3}{\times}{-}\overset{2q}{\bullet} = \overset{-3}{\times}{-}\overset{0}{\bullet}$ is just $\Omega^2_{\mathbb{P}}$. Thus

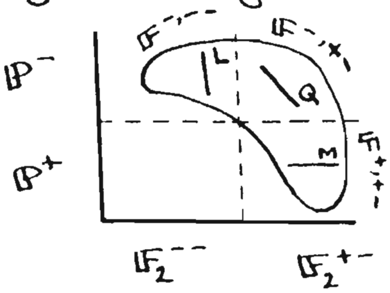
$$\delta(\varphi \cup \overline{\tilde{S}\psi}) := \varphi \cdot \overline{\psi} \in H^2(\mathbb{P}, \Omega^2) \cong \mathbb{C}.$$

(B) In a similar fashion:

$$\tilde{S}: H^1(\mathbb{P}^-, \overset{p}{\times}{-}\overset{q}{\bullet}) \cong H^0(\mathbb{P}^{*+}, \overset{q}{\bullet}{-}\overset{-p-q-3}{\times})$$

which is then conjugate isomorphic with $H^0(\mathbb{P}^+, \overset{-p-q-3}{\times}{-}\overset{q}{\bullet})$. (No quotients by F occur here.) Again, elements pair into $H^2(\mathbb{P}, \overset{-q-3}{\times}{-}\overset{2q}{\bullet})$ which is isomorphic with $\mathbb{C}$ for $q = 0$. The bundles $\overset{p}{\times}{-}\overset{0}{\bullet}$ are the line bundles on $\mathbb{P}$.

§III Using part II, we shall give a twistor transform for discrete series representations. In general, the discrete series occur in the cohomology of (homogeneous) line bundles over the open G-orbits in the full flag variety (in this case, $\mathbb{F}_{12}$). The line bundles should satisfy a negativity requirement (similar to anti-dominance for lowest weights in finite dimensional representations - especially as treated in the Borel-Weil theorem). The flag variety naturally imbeds into the product $\mathbb{F}_2 \times \mathbb{P}$



The open G-orbits in $\mathbb{F}_{12}$ are $\mathbb{F}^{-,--}$, $\mathbb{F}^{-,+-}$ and $\mathbb{F}^{+,+-}$. L, Q and M represent their respective maximal compact subvarieties (which are $\mathbb{CP}^1$'s). Under the natural projections

$$\begin{tikzcd} & \mathbb{F}_2 \times \mathbb{P} \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ \mathbb{F}_2 & & \mathbb{P} \end{tikzcd}$$

$\mathbb{F}^{-,--}$ projects to $\mathbb{F}_2^{--}$ with fiber $\mathbb{CP}^1$ and $\mathbb{F}^{+,+-}$ projects to $\mathbb{P}^+$ with fiber $\mathbb{CP}^1$. $\mathbb{F}^{-,+-}$ is a little tougher to work with, hence interesting. However we will avoid this case. For any subset Y of $\mathbb{P}$, $\mathbb{F}_2$ or $\mathbb{F}$, let $|Y|$ denote its closure and $\overline{Y}$ its complex conjugate. Notice:

$$\overline{\mathbb{F}^{-,--}} = \mathbb{F}^{+,+-} \quad \text{and} \quad \overline{\mathbb{F}^{-,+-}} = \mathbb{F}^{-,+-}.$$

To produce discrete series representations, let's consider

$$H^1(|\mathbb{F}^{-,--}|, \overset{-p}{\times}\!-\!\!\!-\!\overset{-q}{\times}) \quad \text{with } q \geq 0 \text{ and } p \geq 2.$$

Using the fibration π_2 and the earlier isomorphisms we have:

$$H^0(|\mathbb{F}^{-,--}|, \overset{-p}{\times}\!-\!\!\!-\!\overset{-q}{\times}) \cong H^0(|\mathbb{F}_2^{--}|, \overset{p-2}{\bullet}\!-\!\!\!-\!\overset{-p-q+1}{\times})$$

$$\cong H^1(|\mathbb{P}^-|, \overset{q-2}{\times}\!-\!\!\!-\!\overset{p-2}{\bullet})$$

Using π_1:

$$H^1(|\mathbb{P}^-|, \overset{q-2}{\times}\!-\!\!\!-\!\overset{p-2}{\bullet}) \cong H^1(|\mathbb{F} - \mathbb{F}^{+,+-}|, \overset{q-2}{\times}\!-\!\!\!-\!\overset{p-2}{\times}).$$

Putting it all together produces the twistor transform, Mk. II:

$$\boxed{\tilde{\mathcal{S}}: H^1(|\mathbb{F}^{-,--}|, \overset{-p}{\times}\!-\!\!\!-\!\overset{-q}{\times}) \cong H^1(|\mathbb{F} - \mathbb{F}^{+,+-}|, \overset{q-2}{\times}\!-\!\!\!-\!\overset{p-2}{\times})}$$

whence, for $\varphi, \psi \in H^1(|\mathbb{F}^{-,--}|, \overset{-p}{x}\!-\!\!\!-\overset{-q}{x})$ we have $\overline{\tilde{\mathfrak{z}}\psi} \in H^1(|\mathbb{F} - \mathbb{F}^{-,--}|, \overset{p-2}{x}\!-\!\!\!-\overset{q-2}{x})$. Applying the Mayer-Vietoris connecting map to the cup product gives

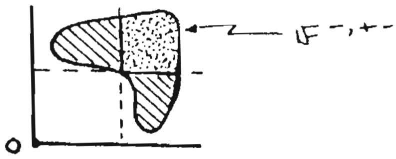
$$\delta(\varphi \cup \overline{\tilde{\mathfrak{z}}\psi}) \in H^3(\mathbb{F}_{12}, \overset{-2}{x}\!-\!\!\!-\overset{-2}{x}) = H^3(\mathbb{F}, \Omega^3) \cong \mathbb{C}.$$

This produces the G-invariant Hermitian pairing necessary for unitarizing the discrete series. Similar pairings exist for the cohomology groups on $\mathbb{F}^{+,+-}$ and $\mathbb{F}^{-,+-}$.

§IV NOTES:

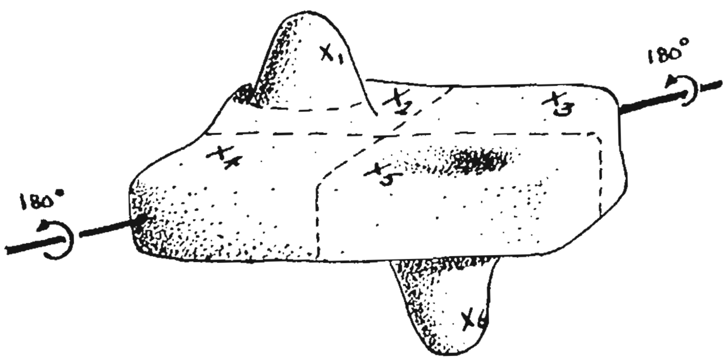
1) The quotient by F occurs only for 'small' groups such as $SU(1,1)$ and $SU(1,2)$.

2) The picture of the isomorphism for $\mathbb{F}^{-,+-}$ is that cohomology on $|\mathbb{F}^{-,+-}|$ is isomorphic to cohomology on $|\mathbb{F} - \mathbb{F}^{-,+-}|$:



Cohomology on the 'dots' $\cong$ cohomology on the 'lines'.

3) For $SU(2,2)$ the picture of $\mathbb{F}_{123}$ in $\mathbb{P} \times \mathbb{F}_2 \times \mathbb{F}_3$ is:



The permutation of the orbits effected by conjugation can be visualized as rotation by 180° about an axis through the orbits X_3 and X_4. (Conjugation on $\mathbb{F}_{12}$ has a similar interpretation) Thus $X_1 \leftrightsquigarrow X_6$ $X_2 \leftrightsquigarrow X_5$ $X_3 \circlearrowleft$ and $X_4 \circlearrowleft$

Again cohomology on the orbit X_i is related to cohomology on the complement of its conjugate, $\overline{X}_i$:

$$H^s(|X_i|, \mathcal{X}) \cong H^d(|X - \overline{X}_i|, \mathcal{X}^\vee).$$

where the degrees are just right to get a pairing into $H^6(\mathbb{F}_{123}, \Omega^{top}) \cong \mathbb{C}$.

4) The general isomorphism (arbitrary groups) can be proved on the level of elementary states (a.k.a. K-types) by comparing a calculation of local cohomology [EJB] with a calculation of formal neighborhoods as in Schmid's thesis. For the isomorphism as representations,

the machinery of $\mathcal{D}$-modules is needed. The twistor transform is then interpretted as a relation between two different types of direct image modules.

Many thanks to RJB.

Mike Eastwood and Ed Dunne.

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A generalised Kerr–Robinson theorem

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Abstract. The Kerr and Robinson theorems in four-dimensional spacetime together provide the general null solution of Maxwell's equations. Robinson's theorem reduces the problem to that of obtaining certain null foliations. The Kerr theorem shows how to represent such foliations in terms of analytic varieties in complex projective 3-space. In this paper we generalise these results to spinor fields of higher valence in spacetimes of arbitrary even dimension. We first review the theory of spinors and twistors for these higher dimensions. We define the appropriate generalisations of Maxwell's equations, and null solutions thereof. It is then proved that the Kerr and Robinson theorems generalise to all even dimensions. We discuss various applications, examples and further generalisations. The generalised Robinson theorem can be seen to extend to curved spaces which admit such null foliations. In the case of Euclidean reality conditions, the generalised Kerr theorem determines all complex structures compatible with the flat metric in terms of freely specified complex analytic varieties in twistor space. Interpretations of the generalised Kerr theorem are also provided for Lorentzian and ultrahyperbolic signatures.

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A conformally invariant connection and the space of leaves of a shear free congruence

Toby Bailey

February 19, 1988

Introduction

This is a report on work in progress, studying the structure of the complex surface which is the space of leaves of a (complexified) shear free congruence. I will show below that in conformal vacuum space-times, the surface has the first formal neighbourhood of an embedding in a complex three manifold (which in the flat space would be dual projective Twistor space).

In order to describe this structure, I will first show that a conformal complex space-time with two spinor fields has a natural conformally invariant connection, which is essentially given by R.P.'s 'conformally invariant edth and thorn operators'. This construction seems to have some geometric interest in its own right.

It is hoped that these these structures will help to explain the separation of various equations in the Kerr metric, and there may be other applications.

The conformally invariant connection

Let $\mathcal{M}$ be a complex conformal space-time, with two independent spinor fields o^A and ι^A, defined up to scale. Equivalently we have a splitting

$$\mathcal{O}^A = O \oplus I \tag{1}$$

of the spin bundle.

Assume also that we are given an identification of the primed and unprimed conformal weights

$$[1] \stackrel{def}{=} \mathcal{O}_{[AB]} \cong \mathcal{O}_{[A'B']}$$

This is equivalent to allowing conformal transformations only of the form

$$\epsilon_{AB} \mapsto \Omega\epsilon_{AB} \qquad\qquad \epsilon_{A'B'} \mapsto \Omega\epsilon_{A'B'}$$

which is a natural condition if $\mathcal{M}$ is the complexification of a real space-time.

Given a metric in the conformal class, the splitting in equation 1 allows us to define a one form

$$Q_a := -2o^{(B}\iota^{C)}\partial_{A'B}(o_{(A}\iota_{C)}) = \rho' l_a + \rho n_a - \tau' m_a - \tau\bar{m}_a$$

where ∂_a is the metric connection, and we adopt the convention that $o_A\iota^A = 1$ whenever a particular metric has been chosen. Under conformal transformatiom

$$Q_a \mapsto Q_a - \Upsilon_a \qquad \text{where} \qquad \Upsilon_a = \Omega^{-1}\partial_a\Omega \tag{2}$$

The significance of Q_a is that it enables us to split the *Local Twistor* bundle as a direct sum.

Recall the Local Twistor exact sequence

$$\begin{array}{ccccccc} 0 \longrightarrow & \mathcal{O}_{A'} & \longrightarrow & \mathcal{O}^\alpha & \longrightarrow & \mathcal{O}^A & \longrightarrow 0 \\ & \pi_{A'} & \longmapsto & (0, \pi_{A'}) & & & \\ & & & (\omega^A, \pi_{A'}) & \longmapsto & \omega^A & \end{array}$$

and the conformal transformation rule

$$\omega^A \mapsto \omega^A \qquad\qquad \pi_{A'} \mapsto \pi_{A'} + i\Upsilon_a\omega^A$$

If we set

$$\alpha_{A'} = \pi_{A'} + iQ_a\omega^A$$

then from equation 2 there is a conformally invariant splitting

$$\begin{array}{ccc} \mathcal{O}^\alpha & \xrightarrow{\cong} & \mathcal{O}^A \oplus \mathcal{O}_{A'} \\ (\omega^A, \pi_{A'}) & \longmapsto & \omega^A \oplus \alpha_{A'} \end{array} \tag{3}$$

of $\mathcal{O}^\alpha$ and I will use the 'split co-ordinates' $(\omega^A, \alpha_{A'})$ henceforth.

The *Local Twistor connection* splits to give connections, which I will denote by ∇_a, on the various spin bundles. A brief calculation shows these to be

$$\begin{array}{lcl} \mathcal{O}^A & : & \nabla_b\mu^A = \partial_b\mu^A + \epsilon_B{}^A Q_{CB'}\mu^C \\ \mathcal{O}^{A'} & : & \nabla_b\mu^{A'} = \partial_b\mu^{A'} + \epsilon_{B'}{}^{A'} Q_{BC'}\mu^{C'} \\ \mathcal{O}_A & : & \nabla_b\mu_A = \partial_b\mu_A - Q_{AB'}\mu_B \\ \mathcal{O}_{A'} & : & \nabla_b\mu_{A'} = \partial_b\mu_{A'} - Q_{BA'}\mu_{B'} \\ \mathcal{O}_{[AC]} & : & \nabla_b\nu_{AC} = \partial_b\nu_{AC} - Q_b\nu_{AC} \end{array}$$

If $Z^\alpha = (\omega^A, \alpha_{A'})$ is a local twistor, we can write the Local Twistor connection as

$$\nabla_b Z^\alpha = (\nabla_b\omega^A + i\epsilon_B{}^A\alpha_{B'}, \nabla_b\alpha_{A'} + iD_{ab}\omega^A) \tag{4}$$

where D_{ab} is a conformally invariant modification of P_{ab} defined by

$$D_{ab} = P_{ab} - \partial_b Q_a + Q_{AB'}Q_{BA'}$$

The splitting in equation 1 allows us to define the bundles

$$\langle -r', -r\rangle := O^{r'} \otimes I^r$$

(note that $\langle 1,1\rangle = [1]$). The connection ∇_a can be projected on to these. For example, if λ^A is a section of $\langle -1, 0\rangle$, so that $\lambda^A o_A = 0$,

$$\lambda^A \longmapsto -o^A\iota_C\nabla_b\lambda^C$$

is a connection, and its components are given by 'conformally invariant edth and thorn', in just the same way as the same expression with the metric connection ∂_b replacing ∇_b has components that can be computed with ordinary edth and thorn.

Since ∇_a agrees with ∂_a if you form any of the well known conformally invariant parts of the metric connection, there is scope here for producing a complete 'conformally invariant G.H.P. formalism'. The expressions which arise as curvatures when one commutes conformal edths and thorns are components of D_{ab}. The geometrical significance of these connections will be discussed in a later section.

Shear free congruences in Minkowski space

Before starting on the general case, I will review the situation in flat space-time. In real Minkowski space, a shear free congruence of null geodesics (hereafter SFR) is given by a spinor field satisfying

$$o^A o^B \partial_a o_B = 0 \tag{5}$$

If o_A is analytic, it can be complexified, and it then determines a distribution of β-planes. This distribution is integrable, and so gives a foliation of Minkowski space by complex surfaces precisely when o_A is shear free. The space of leaves S of this foliation is the hypersurface in dual projective Twistor space $\mathbf{P}^*$, which describes the congruence, according to Kerr's Theorem.

The surface S inherits some structure from its embedding, in particular there is the tangent bundle of $\mathbf{P}^*$ which sits in the *normal bundle sequence* and the restrictions of the line bundles $\mathcal{O}(n)$. The analysis in the accompanying article in this T.N. shows how massless fields of various orders along the congruence are isomorphic to sections of sheaves on S. I will now describe how this generalises to curved space.

SFRs in curved space-times

In a general space-time, an SFR is still given by a solution of equation 5, and gives a foliation in the complexification. The space of leaves still defines a complex surface S, but there is in general no Twistor space in which it is embedded.

The SFR defines a Maxwell field, which in Minkowski space is the Ward transform of the line bundle defined by S considered as a divisor. This follows from the fact that equation 5 is equivalent to the existence of a one form Φ_a with

$$\partial_{A'(A} o_{B)} = \Phi_{A'(A} o_{B)}$$

and it is easy to see that Φ_a has precisely the freedom to be the potential for a Maxwell field. The left handed part $\phi_{AB} = \partial_{A'(A}\Phi^{A'}_{B)}$ satisfies $\Psi_{ABCD}o^D = -\phi_{(AB}o_{C)}$, and so vanishes as expected in a conformally flat space- time[1].

[1]An SFR is thus a charged Twistor coupled to its own canonically defined Maxwell field

The structures I shall describe on S only exist under certain conditions. In particular, I will say that the SFR o_A in the space-time $\mathcal{M}$ satisfies the *Goldberg–Sachs condition* (hereafter GS) if

$$o^A o^B o^C \Psi_{ABCD} = 0$$

We assume the GS condition holds henceforth, since no significant part of the structure on S seems to exist otherwise. The Goldberg–Sachs Theorem implies that the GS condition is equivalent to $o^A o^B o^C \partial^D_{D'} \Psi_{ABCD} = 0$ and it is therefore satisfied by all conformally vacuum space-times.

To construct bundles on S, we make use of ∇_a, the conformally invariant connection. First choose a spinor direction ι_A to complement the SFR o_A, and deduce from the SFR and GS conditions that on all the bundles $\langle r', r\rangle$ and $\mathcal{O}^{A'}$, the part $o^A\nabla_a$ of the connection that differentiates up the leaves of the foliation is both independent of the choice of ι_A and flat.[2]

We can thus define line bundles $\langle r', r\rangle_S$ and a rank two vector bundle $\mathcal{O}(S)^{A'}$ over S, whose sections are ***by definition*** sections of the corresponding bundle on $\mathcal{M}$ with vanishing conformal derivitive up the foliation.

The dual Local Twistor bundle also defines a vector bundle on S. We have an injection of the spinors proportional to o_A into $\mathcal{O}_\alpha$

$$0 \longrightarrow \langle 0, 1\rangle \longrightarrow \mathcal{O}_\alpha \longrightarrow E \longrightarrow 0$$

defining the quotient E. The part $o^A\nabla_a$ of the Local Twistor connection preserves $\langle 0, 1\rangle$ and hence is well defined on E. Furthermore, it is flat on the leaves and so defines a rank three vector bundle $\mathcal{E}$ on S.

Sections of E can be realised as spinor fields $\xi^{A'}$ satisfying a tangential Twistor equation[3]

$$o^A \nabla_A{}^{(A'}\xi^{B')} = 0$$

and given that sections of $\mathcal{O}(S)^{A'}$ are spinor fields satisfying

$$o^A \nabla_{AA'}\xi^{B'} = 0$$

[2]It is helpful to note that $o^A Q_a$ is independent of ι_A if o^A is SFR.

[3]to see this, note that GS and SFR imply $o^A o^B D_{ab} = 0$ and use the conjugate version of equation 4. When writing down the splitting and connection on the dual Local Twistors, simply write down the conjugate *pretending that D_{ab} and Q_a are real.*

we get an injection $\mathcal{O}(S)^{A'} \to \mathcal{E}$ which extends to give a short exact sequence

$$0 \longrightarrow \mathcal{O}(S)^{A'} \longrightarrow \mathcal{E} \longrightarrow \langle 1,0\rangle \longrightarrow 0$$

given, in terms of equations, by

$$\begin{array}{ccccc} o^A\nabla_{AA'}\xi^{B'} = 0 & \longmapsto & o^A\nabla_A{}^{(A'}\xi^{B')} = 0 & \longmapsto & \begin{pmatrix} o^A o^B \nabla_{BB'}\eta_A = 0 \\ \iota^A\eta_A = 0 \end{pmatrix} \\ & & \xi^{A'} & \longmapsto & \iota_A o^B \nabla_{BB'}\xi^{B'} \end{array}$$

If $\mu^{A'}$ is a section of $\mathcal{O}^{A'}\langle 0,-1\rangle$, a calculation reveals that the condition $o^A\nabla_a\mu^{B'} = 0$ is what is required[4] to make $\iota^A\mu^{A'}$ a connecting vector to a nearby leaf of the foliation. Thus, $\mathcal{O}(S)^{A'}\langle 0,-1\rangle_S$ can be identified with the tangent bundle $T(S)$ of S. The exact sequence above can be tensored through by $\langle 0,-1\rangle_S$ to give what in flat space would be the *normal bundle sequence* of S

$$0 \longrightarrow T(S) \longrightarrow \mathcal{E}\langle 0,-1\rangle_S \longrightarrow \langle 1,-1\rangle_S \longrightarrow 0$$

If one is given a hypersurface in a complex manifold, then knowing the normal bundle sequence is equivalent to knowing the *first formal neighbourhood* of the embedding. I will now briefly describe how one can realise the first formal neighbourhood of an embedding of S directly.

The spinor field o_A defines a natural embedding of the space-time $\mathcal{M}$ in the *projective spin bundle* $\mathbf{P}\mathcal{O}_A$. Now realise S by choosing a two-surface $\tilde{S}$ transverse to the foliation, and note that $\tilde{S}$ has a natural embedding in the restriction of $\mathbf{P}\mathcal{O}_A$. The *first formal neighbourhood* of this embedding is independent of the choice of $\tilde{S}$, and so defines a first formal neighbourhood sheaf $\mathcal{O}^{(1)}$ on S.

In slightly more detail; recall that $\mathbf{P}\mathcal{O}_A$ has a naturally defined differential operator $\pi^A\partial_a$ which defines a two-plane distribution, the integral surfaces of which (if it has any) are lifts of β-surfaces.

When o_A is an SFR, there is a two complex parameter family of β-surfaces parametrised by S, and functions f on $\mathbf{P}\mathcal{O}_A$ which obey $\pi^A\partial_a f = 0$ *on the lift of* $\mathcal{M}$ are precisely functions on S.

[4] The connection here is the tensor product of the conformally invariant ones on the factors

A calculation shows that, given the GS condition, there are two functions of two complex variables worth of functions g on $\mathbf{P}\mathcal{O}_A$ that obey $\pi^A \partial_a g = 0$ *to first order in a neighbourhood of the lift of* $\mathcal{M}$. These form the formal neighbourhood sheaf $\mathcal{O}^{(1)}$ on S.

In terms of the conformally invariant connections, a function on the first formal neighbourhood of the lift of $\mathcal{M}$ can be written

$$g(x, \pi_A) = f(x) + \iota^A \chi_A{}^B \pi_B \qquad \text{where} \qquad o^A \chi_A{}^B = 0 = \chi_A{}^B o_B$$

If the spinor field $\chi_A{}^B$ satisfies

$$\nabla_{BA'} \chi_A{}^B = \nabla_{AA'} f$$

then it defines a section of $\mathcal{O}^{(1)}$.

Massless fields

One result of this analysis is a minor generalisation of Robinson's Theorem, which states that if o_A is an SFR, then, for each helicity, there are precisely one holomorphic function of two complex variables worth of left handed massless fields null along it. If the field has n indicees, then remembering that it has conformal weight -1, it is easy to check that these fields are in one to one correspondence with sections over S of $\langle 1, n+1 \rangle_S$.

In my accompanying article I show how *in flat space* fields of various orders along o_A correspond to sections of sheaves over S. Provided, as usual, that the GS condition holds, it turns out that sections of the formal neighbourhood sheaf $\mathcal{O}^{(1)} \otimes \langle 1, 1 \rangle_S$ on S do give left handed Maxwell fields which have a principal null direction along the congruence. Thus there are two holomorphic functions of two complex variables worth of such things, just as in the flat case.

Apart from that case however, more severe curvature restrictions appear. To get three functions worth of order three Maxwell fields one requires $o^A o^B \Psi_{ABCD} = 0$ in which case it seems that S has a second formal neighbourhood sheaf.

Killing spinors

Suppose $\mathcal{M}$ admits a *Killing spinor*, and choose o_A and ι_A to be along its principal null directions. The Killing spinor equation

$$\partial_{A'}{}^{(A}\omega^{BC)} = 0$$

then implies that both o_A and ι_A are SFRs. The remaining parts of the equation reduce to solving $\nabla_a\omega = 0$ where ω is a section of $\langle 1,1\rangle$. This is only possible if the conformally invariant connection on $\langle 1,1\rangle$ is flat, which implies

$$\partial_{[a}Q_{b]} = 0$$

This has a number of consequences. Firstly, it provides an isomorphism $\langle 1,1\rangle \cong \langle 0,0\rangle$ which carries over to S, thereby giving a natural trivialisation of $\langle 1,1\rangle_S$. Secondly, the fact that Q_a is closed means that locally it is exact, and equation 2 shows that it can thus be made to vanish by a conformal transformation. In the special metric thus constructed, *all* the curvature information is contained in the single line bundle $\langle 1,0\rangle$ and its (conformally invariant) connection[5]. Further work is in progress on all this, since it seems likely that, combined with the ideas in the next section, it will be possible to explain the separation of various differential equations in the Kerr solution.

Geometrical significance

To finish, I will mention some ideas due to R.P. and K.P.T. which I have just started to follow up in collaberation with M.A.S. The conformally invariant connection constructed above is an example of a unique connection determined by a geometrical structure, and the structure one has (in the complex space- time) seems to be that which would be obtained on the complexification of a real four manifold X with an almost complex structure $J_a{}^b$ and a compatible conformal Hermitian metric. The eigenspaces of $J_a{}^b$ are the two-plane distributions defined by o_A and ι_A so that

$$J_a{}^b = i(o_A\iota^B + \iota_A o^B)\epsilon_{A'}{}^{B'}$$

The almost complex structure will be integrable when both o_A and ι_A are SFRs. Further, the suggestion is that the existence of a Killing spinor

[5] c. f. B.P.J. in Proc. Roy. Soc. A392 p323–341 (1984)

is equivalent to the Kähler condition on the Hermitian metric. This seems very likely since something very similar has been given by Flaherty[6], whose view-point is somewhat different.

I would like to thank M.A.S., R.P., and K.P.T. for discussions and suggestions.

[6]Hermitian and Kählerian geometry in relativity. Lecure Notes in Physics 46 (1976)

Relative cohomology power series, Robinson's Theorem and multipole expansions

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Introduction

In my original articles on the Twistor description of fields with sources on a world-line[1] I gave some expressions for "multipoles" based on a world-line. In this note, I will show how a first cohomology class, relative to a hypersurface, can be expanded in a sort of "power series", which seems to be the Twistor version of the multipole expansion. The power series also gives a precise "abstract nonsense" version of the Twistor description of algebraically special fields.

The relative cohomology power series

Let S be a hypersurface in a complex manifold X, and let $\mathcal{F}$ be a locally free sheaf of $\mathcal{O}_X$ modules on X. The *relative cohomology group* $H^1_S(X, \mathcal{F})$ can be described by a relative Čech cocycle, but a good intuitive picture is as follows: Choose an open cover U_i of a neighbourhood of S in X; then a representative is given by a set f_i of sections of $\mathcal{F}$ over U_i that 'blow up' on S, with the restriction that $f_i - f_j$ is holomorphic on *all* of $U_i \cap U_j$. The freedom in each f_i is the addition of a holomorphic section of $\mathcal{F}$.

[1] T.N. 14,15 and Proc. Roy. Soc. A397 143–155 (1985)

Now let g_i be *defining functions* for S, then one might try and expand the relative class defined by the f_i as a power series

$$f_i = \frac{f_i^{(1)}}{g_i} + \frac{f_i^{(2)}}{g_i^2} + \cdots \frac{f_i^{(n)}}{g_i^n} + \cdots \tag{1}$$

To understand this we need the *divisor bundle* L of S, which is defined to be the line bundle with transition functions g_i/g_j on $U_i \cap U_j$. The functions g_i then give a *distinguished section* s of L which has a simple zero on S. The section s gives us a map

$$s^k : \mathcal{F} \longrightarrow \mathcal{F} \otimes L^k$$

which induces a map on the relative cohomology.

Definition 1 *The k-th order relative cohomology $H^1_S(X, \mathcal{F}; k)$ is defined by the exactness of*

$$0 \longrightarrow H^1_S(X, \mathcal{F}; k) \longrightarrow H^1_S(X, \mathcal{F}) \xrightarrow{\times s^k} H^1_S(X, \mathcal{F} \otimes L^k)$$

The k-th order cohomology is thus the part which has a pole of order k or less on S, and it therefore corresponds to the first k terms in equation 1 above.

If $\mathcal{E}$ is a sheaf on X, and $\mathcal{I}^{(p)}\mathcal{E}$ is the ideal of sections of $\mathcal{E}$ which vanish to p-th order on S we can define the *p-th formal neighbourhood sheaf* $(\mathcal{E})^{(p)}$ by the short exact sequence

$$0 \longrightarrow \mathcal{I}^{(p+1)}\mathcal{E} \longrightarrow \mathcal{E} \longrightarrow (\mathcal{E})^{(p)} \longrightarrow 0 \tag{2}$$

so that $(\mathcal{E})^{(0)}$ is just $\mathcal{E}$ restricted to S.

Lemma 1 *There is a natural isomorphism*

$$H^1_S(X, \mathcal{F}; k) \cong \Gamma(S, (\mathcal{F} \otimes L^k)^{(k-1)})$$

The proof is simply to observe that in equation 1 above, the $f_i^{(k)}$ must give a section of $\mathcal{F} \otimes L^k$ with the freedom as given by equation 2.

Thus we have strictly a *filtration* of the relative cohomology (rather than an infinite direct sum), with the quotient at each stage given by the exact sequence

$$0 \longrightarrow \Gamma(S, (\mathcal{F} \otimes L^{k-1})^{(k-2)}) \xrightarrow{\times s} \Gamma(S, (\mathcal{F} \otimes L^k)^{(k-1)} \longrightarrow \Gamma(S, \mathcal{F} \otimes L^k) \longrightarrow 0$$

Algebraically special fields

The above analysis can be applied when S is a hypersurface in a region X in projective Twistor space, corresponding to a shear free congruence. We can define cohomology of order k on S just as for the relative case, and we will say that a right handed massless field is of order k on the congruence if its Twistor function is in $H^1(X, \mathcal{O}(-n-2); k)$. Thus, order 1 means null, order n means the field has a pnd. along the congruence, and higher orders correspond to certain differential relations between the field and the congruence.

If one writes down the commutative diagram whose rows are the relative cohomology sequences, and whose columns are induced by $s^k : \mathcal{F} \to \mathcal{F} \otimes L^k$, it is easy to see that if $H^1(X, \mathcal{F}) = 0$ then there is an exact sequence

$$0 \longrightarrow \frac{\Gamma(X, \mathcal{F} \otimes L^k)}{\Gamma(X, \mathcal{F})} \longrightarrow H^1_S(X, \mathcal{F}; k) \longrightarrow H^1(X, \mathcal{F}; k) \longrightarrow 0$$

Since L has the section s which has a simple zero on S, which intersects every line in X exactly once, we can write $L = M(1)$ where M is a line bundle trivial on every line in X[2]. Thus if $k < n+2$

$$\Gamma(X, L^k(-n-2)) = \Gamma(X, M^k(k-n-2)) = 0$$

and so

$$H^1_S(X, \mathcal{O}(-n-2); k) \cong H^1(X, \mathcal{O}(-n-2); k); \quad k < n+2$$

The result of all this is a statement of the (generalised) flat space Robinson Theorem: The space of helicity $n/2$ right handed massless fields of order k ($k < n+2$) along the congruence is isomorphic to $\Gamma(S, (L^k(-n-2))^{(k-1)})$. This is precisely the 'k holomorphic functions of two complex variables' described by R.P. and W.R. in SS-T. (Vol. 2 p. 206).

The particular case where $k = 1$ and $n = 2$ was examined by M.G.E. (T.N.20 p. 31). We get that these fields are given by sections of $L(-4)$ over S, but $\mathcal{O}(-4) = \Omega^3$ and L restricted to S is just the normal bundle. Thus

[2] M is the Ward bundle of the 'Maxwell field of the congruence' — see my accompanying article

$L(-4) = \Omega_S^2$, we get an isomorphism of the null right handed Maxwell fields with holomorphic 2-forms on S.

The null Maxwell fields inject into the order 2 fields, and give a quotient sheaf

$$0 \longrightarrow \Gamma(S, L(-4)) \longrightarrow \Gamma(S, (L^2(-4))^{(1)}) \longrightarrow \Gamma(S, L^2(-4)) \longrightarrow 0$$

The quotient corresponds in space-time to neutrino fields of order 1, coupled to the Maxwell field of the congruence. The map onto this group is 'helicity lowering', where the congruence is regarded as a charged Twistor.

Multipole expansions

If S is the *ruled surface* corresponding to a world-line in Minkowski space, the first relative cohomology describes massess fields with sources on the world-line[3]. We can use the analysis given above to get a filtration of these fields.

It seems that the first terms (eg. order 2 for right handed Maxwell and order 3 for right handed gravity) give the fields with non-vanishing 'charges',and the remainder give an expansion in 'multipoles' where, for example, a 2^p- pole for a right handed helicity $n/2$ field is given by

$$\phi_{\underbrace{A'\ldots K'}_{n}} = \oint \sigma^{\overbrace{A\ldots N}^{n+p}\overbrace{P\ldots S}^{p}} \dot{y}_P^{L'} \ldots \dot{y}_S^{N'} \nabla_{AA'} \ldots \nabla_{NN'} \frac{ds}{(x - y(s))^2}$$

where $\sigma^{A\ldots S}$ is a totally symmetric spinor function of s, the proper time along the world-line $y^a(s)$. Under conformal transformations, a 2^p-pole gets mixed with lower ordered terms, which is what one might expect given that the Twistor space expansion is not a direct sum.

There are still many details to be tidied up here, and further work is in progress.

I am very grateful to M.A.S. for discussions about this work.

[3] and the Maxwell field of the congruence is the left handed part of the field of a unit charge on the world-line

Abstract

The Geroch group and non-Hausdorff twistor spaces

N.M.J. Woodhouse & L.J. Mason

By reducing the Ward correspondence, we show that there is a correspondence between stationary axisymmetric solutions of the vacuum Einstein equations and a class of holomorphic vector bundles over a reduced twistor space, which is a compact one-dimensional, but non-Hausdorff, complex manifold. We show that the solutions generated by Ward's ansätze correspond to bundles which have a simple behaviour on the 'real axis' in the reduced space. We identify the Geroch group (Kinnersley and Chitre's 'group K') with a subgroup of the loop group of GL(2,$\mathbb{C}$) and we describe its orbits. We also identify some of the subgroups which preserve asymptotic flatness.

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