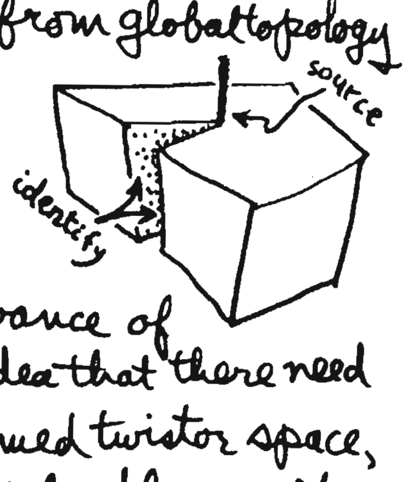
Topological QFT and Twistors: Holomorphic Linking

There is much current interest in the idea of "topological quantum field theories" (TQFT) particularly in relation to pure mathematical applications: Fleur cohomology, Donaldson invariants, Jones polynomials, etc. However the physical relevance of such "field theories" is more problematical. The salient feature of a TQFT is that there are no field equations — the dynamics vanishes altogether locally, and the theory can be formulated to be independent of the choice of metric. This is very attractive from the mathematical point of view — where one may be interested in global (differential) topological properties rather than properties depending upon a chosen metric or (physical) field. But physical relevance remains unclear. A noteworthy example, where field equations vanish locally, is provided by (2+1)-dimensional "gravity": in "vacuum" we have $R_{ab}=0$ whereas the Weyl curvature necessarily vanishes ($C_{abcd}=0$) in 3-dimensions [cf. E. Witten preprint "2+1 Dimensional Gravity as an Exactly Soluble System", Sept. 1988, IASSNS-HEP-88/32]. Thus the "vacuum space-time" is always flat ($R_{abcd}=0$), the (classical) interest coming from global topology and "deficit angles" as regions of "non-vacuum" (singular world-lines?) are circumnavigated.



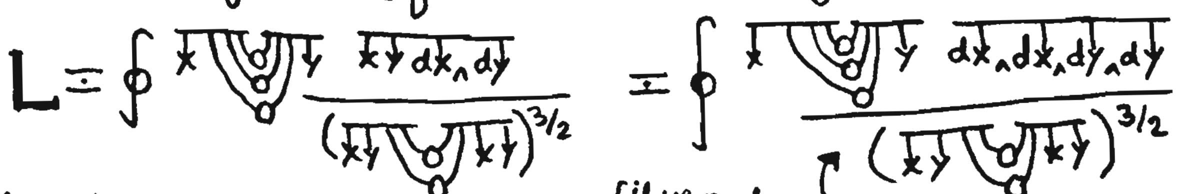
Twistor theory offers a very different viewpoint as to the possible physical relevance of a TQFT. We are familiar with the idea that there need be no local information stored in a deformed twistor space, or in a twistor space with a holomorphic bundle over it, or a 1-function (element of an H^1) defined on it. Thus, the physically appropriate home for TQFT might well be twistor space rather than space-time.

There is, of course, one fundamental difference between the envisaged setting for TQFT and twistor space: TQFTs refer to <u>real</u> manifolds whereas twistor spaces are <u>complex</u>. However, this need not be a drawback. Ideas can be carried over from the real to the (holomorphic-) complex whenever there is an <u>analytic integral formula expressing</u> what is needed. This can lead to some surprising applications.

Consider the ordinary <u>linking number</u> $\mathbf{L}$ for two closed curves in Euclidean 3-space. There is a formula (cf. Flanders, "Differential Forms", p. 80)

$$\mathbf{L} = \frac{1}{4\pi}\oint\!\!\oint \frac{(\vec{x}-\vec{y})\cdot d\vec{x}\wedge d\vec{y}}{|\vec{x}-\vec{y}|^3}$$

for this (expressing the work done on a magnetic monopole carried around one curve, where there is a fixed current in the other — or, no more physically, the work done on an electric charge carried around a "magnetic current"!). We want to interpret this formula as a complex contour integral over (contours in) two holomorphic curves in $\mathbb{CP}^3$, i.e. in $\mathbb{PT}$. (N.B. curves in $\mathbb{RP}^3$ can have quarter-integral linking numbers; but the following <u>might</u> encounter unwanted branching in the $(\ldots)^{3/2}$ in those cases.) We can rewrite $\mathbf{L}$ <u>projectively</u> (using the sign $\doteq$ to denote equality up to a multiplicative factor that I haven't yet bothered to work out) as



[if we prefer a non-projective version]

where x and y are twistors specifying points on the two curves X and Y (not necessarily compact), respectively, each varying along its appropriate (1-real-dim. closed) contour, and where γ specifies an arbitrarily chosen "metric" on $\mathbb{PT}$, either complex Euclidean, in which case (= 24 det(γ)) vanishes, or complexified sphere metric (det(γ) $\neq 0$). This expression for $\mathbf{L}$ is independent of small finite deformations of

(a) the contours (obviously)
(b) the curves X and Y (not so ")
(c) the "metric" ⊍ (not so ").

These variations can take place within some neighbourhood $\mathcal{N}$ (a complex manifold) of the contours in $\mathbb{PT}$. We must make sure that the "distance" between x and y never becomes zero (i.e. $x y \neq 0$) within $\mathcal{N}$ (or, rather, on the contours), in our variation.

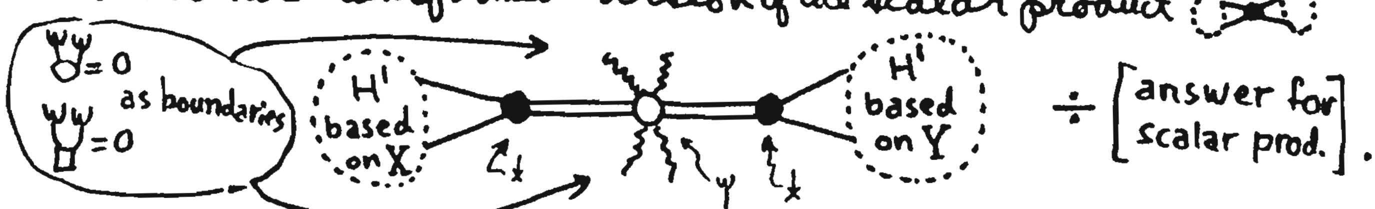
Note the curious fact that this "linking" between X and Y is not simply a topological property since real 2-surfaces in real 6-space don't link. It only works because we are allowing only holomorphic variations of X, Y, ⊍.

The following expression, for which I am grateful for important assistance from A.P.H. → $\dfrac{x\,y}{(xy\,xy)^{3/2}} = \oint \dfrac{\mathcal{D}W \log[\,\Psi\Psi/\Psi\Psi\,]}{(\Psi_x \Psi_y)^2}$ (⊔ "arbitrary") (introduce new ψ) can be used to express **L** as:

$$\mathbf{L} = \oint_{(\text{proj.})} \frac{\log[\,\Psi\Psi/\Psi\Psi\,]\ \psi\, d\psi \wedge d\psi \wedge d\psi \ xy \wedge dx \wedge dy}{(\Psi_x \Psi_y)^2} = \oint_{(\text{non-proj.})} \frac{\log[\,\Psi\Psi/\Psi\Psi\,]\ d^4\psi \wedge dx \wedge dx \wedge dy \wedge dy}{(\Psi_x \Psi_y)^2}$$

which is handy for examining its invariance properties.

A.P.H. points out that this expression is essentially a "double twistor transformed" version of the scalar product



Here the 1-functions are standard cohomology elements specifying the two curves X, Y (cf. R.P. in TN 4; M.G.E. & L.P.H. in TN 8; both reprinted in Adv. in Tw. Th., for the case of a twisted cubic). M.F.A. recently suggested to me a possible role, in this context, for such scalar product expressions (related to Green's functions), and I am grateful to him for valuable conversations.

Thanks are due also to A.P.H. and other members of the twistor group.

QUESTION: can causality relations between points in $\mathbb{M}$ be understood in terms of holomorphic linking of the corresponding $\mathbb{CP}$'s in $\mathbb{PT}$? How does this relate to Rob Low's thesis work?

~ Roger Penrose