Topological QFT and Twistors: Holomorphic Linking

There is much current interest in the idea of "topological quantum field theories" (TQFT) particularly in relation to pure mathematical applications: Floer cohomology, Donaldson invariants, Jones polynomials, etc. However the physical relevance of such "field theories" is more problematical. The salient feature of a TQFT is that there are no field equations — the dynamics vanishes altogether locally, and the theory can be formulated to be independent of the choice of metric. This is very attractive from the mathematical point of view — where one may be interested in global (differential) topological properties rather than properties depending upon a chosen metric or (physical) field. But physical relevance remains unclear. A noteworthy example, where field equations vanish locally, is provided by \((2+1)\)-dimensional gravity in "vacuum" we have \(\mathcal{R}_{\alpha\beta} = 0\) whereas the Weyl curvature necessarily vanishes \((\mathcal{C}_{\alpha\beta\gamma\delta} = 0)\) in 3-dimensional [cf. Witten preprint "\(2+1\) Dimensional Gravity as an Exactly Solvable System", Sept 1988, IASSNS-HEP-88/32]. Thus the "vacuum space-time" is always flat \((\mathcal{R}_{\alpha\beta\gamma\delta} = 0)\), the classical interest coming from global topology and "defect angles" as regions of "non-vacuum" (singular world-lines?) are circumnavigated.

Twistor theory offers a very different viewpoint as to the possible physical relevance of a TQFT. We are familiar with the idea that there need be no local information stored in a deformed twistor space, or in a twistor space with a holomorphic bundle over it, or a \(1\)-function \((\text{element of } \mathbb{H}^1)\) defined on it. Thus, the physically appropriate home for TQFT might well be twistor space rather than space-time.
There is, of course, one fundamental difference between the envisaged setting for TQFT and twistor space: TQFTs refer to real manifolds whereas twistor spaces are complex. However, this need not be a drawback. Ideas can be carried over from the real to the (holomorphic) complex whenever there is an analytic integral formula expressing what is needed. This can lead to some surprising applications.

Consider the ordinary linking number $L$ for two closed curves in Euclidean 3-space. There is a formula (cf. Flanders, "Differential Forms", p. 80)

$$L = \frac{1}{4\pi} \oint \frac{(\mathbf{x} - \mathbf{y}) \cdot d\mathbf{x} \wedge d\mathbf{y}}{|\mathbf{x} - \mathbf{y}|^3}$$

for this (expressing the work done on a magnetic monopole carried around one curve, where there is a fixed current in the other — or, no more physically, the work done on an electric charge carried around a "magnetic current"). We want to interpret this formula as a complex contour integral over (contrary to imaginary) two holomorphic curves in CP³, i.e. in PT. (N.B. Curves in RP³ can have quarter-integral linking numbers; but the following might encounter unwanted branching in the (...)³/2 in those cases.) We can rewrite $L$ projectively (using the sign $\equiv$ to denote equality up to a multiplicative factor that I haven’t yet bothered to work out) as

$$L = \oint \frac{\mathbf{x} \wedge d\mathbf{x} \wedge d\mathbf{y}}{|(\mathbf{x} \wedge \mathbf{y})|^{3/2}}$$

where $\mathbf{x}$ and $\mathbf{y}$ are twistor spaces specifying points on the two curves $X$ and $Y$ (not necessarily compact), respectively, each varying along its appropriate (1-real-dim. closed) contour, and where $\mathbf{y}$ specifies an arbitrarily chosen "metric" on PT, either complex Euclidean, in which case $\mathbf{y} = 24 \det(\mathbf{y})$ vanishes, or complexified sphere metric ($\det(\mathbf{y})^2$). This expression for $L$ is independent of small finite deformations of
(a) the contours (obviously)
(b) the curves \(X\) and \(Y\) (not so ...)
(c) the "metric" \(g\) (not so ...)

These variations can take place within some neighborhood \(\mathcal{N}\) (a complex manifold) of the contours in \(PT\). We must make sure that the "distance" between \(X\) and \(Y\) never becomes zero (i.e., \(\Sigma \ll \infty\)) within \(\mathcal{N}\) (or rather, on the contours) in our variation.

Note the curious fact that this "linking" between \(X\) and \(Y\) is not simply a topological property since real 2-surfaces in real 6-space don't link. It only works because we are allowing only holomorphic variations of \(X, Y, g\).

The following expression, for which I am grateful for important assistance from APH:$$L = \frac{1}{2} \int \frac{2W \log \left[ \frac{1}{w} \right]}{(x^\gamma x^\gamma)^2}$$
(introduce new \(w\))

\(L\) can be used to express \(L\) as:

$$L = \int \frac{\log \left[ \frac{1}{w^\gamma} \right]}{(x^\gamma x^\gamma)^2}$$

which is handy for examining its invariance properties.

APH. points out that this expression is essentially a "double twistor transformed" version of the scalar product $$\cdots$$

Here the \(1\) functions are standard cohomology elements specifying the two curves \(X, Y\) (cf. R.P. in TN 4; M.G.E. & L.P.H. in TN 8; both reprinted in Adv. in Tw.Ts for the case of a twisted cubic).

M.F.A. recently suggested to me a possible role, in this context, for such scalar product expressions (related to Green's functions), and I am grateful to him for valuable conversations.

Thanks are due also to APH. and other members of the twistor group.

**Question**: can causality relations between points in \(\mathcal{N}\) be understood in terms of holomorphic linking of the corresponding CP's in \(PT\)? How does this relate to Rob Low's thesis work? 

~Roger Penrose~