CAUSAL RELATIONS AND LINKING IN TWISTOR SPACE

This is a brief review of some work on the relationship between the causal separation of points (mainly in Minkowski space) and the way in which their corresponding skies in (projective null) twistor space link. To fix notation, let M be Minkowski space, and let PN^I be projective null twistor space without I, so that PN^I is the space of null geodesics of M, with topology $\mathbb{R}^3 \times S^2$. If x is a point in M, then $X \cong S^2$ is its sky in PN^I.

Then let $x,y \in M$, and let S be a Cauchy surface containing y (for example, the surface of constant t). Given this, $PN^I \cong S \times S^2$, and so $PN^I \backslash I$ is diffeomorphic with $(S\backslash\{y\}) \times S^2$, which has the topology $(\mathbb{R} \times S^2) \times S^2$. A simple calculation using the Künneth sequence[1] shows that

$$\begin{aligned} H_2(PN^I\backslash Y) &\cong H_2(S\backslash\{y\}) \oplus H_2(PN^I) \\ &\cong \mathbb{Z} \quad \oplus \quad \mathbb{Z} \end{aligned}$$

(where all homology is taken to be with coefficients in $\mathbb{Z}$). Thinking geometrically, then, if we consider the image of the fundamental class of X in $H_2(PN^I\backslash Y)$, we get a pair of integers, the first of which tells us how often X wraps round Y; this is used to motivate a definition of linking number for skies. First, however, the question of the orientation on a sky must be considered. To any sky in PN^I, associate the orientation which induces the inward pointing normal to $I^-(x) \cap S$ when S is a Cauchy surface just to the past of x.

Definition. Let X and Y be the skies of points of M such that $X \cap Y = \emptyset$. Then Link(X,Y) is defined to be the image of the fundamental class of X in $H_2(PN^I\backslash Y)/H_2(PN^I)$.

It follows that x and y are timelike separated if and only if $\mathrm{Link}(X,Y) \neq 0$. In fact, we have

Theorem. Let $x,y \in M$ with skies $X,Y \subset PN^I$ respectively, such that $X \cap Y = \emptyset$. Then

$$x \in I^{\pm}(y) \text{ if } \mathrm{Link}(X,Y) = \pm 1$$

$$x \in M\backslash J(y) \text{ if } \mathrm{Link}(X,Y) = 0. \qquad \square$$

Given that this version of linking seems to work so well, one is tempted to try to get the intersection theoretic version[2] of linking to work; in this case, one is immediately faced with the problem that the sky of a point is not the boundary of any surface in PN^I, so there is a problem caused by the topological non-triviality of PN^I. One attempt at circumventing this problem might be to consider surfaces in $PN\setminus Y$ with boundary $I\cup X$, corresponding to considering the homology of X in $PN\setminus Y$ relative to I. This cannot work, though, because these relative homology groups all vanish.

A slight subtlety involving the way in which the surface approaches X and I will recover the situation.

Theorem. Let $x,y\in M$ such that $X\cap Y=\emptyset$. Then if $\Sigma\subset PN$ is a surface with boundary $X\cup I$ such that

1) $\Sigma \cong S^2\times[0,1]$

2) $\Sigma\setminus I \cong S^2\times[0,1)$

and 3) Carrying the orientation from X to I along Σ gives the same orientation on I as that induced by completing a Cauchy surface of M and regarding PN as the tangent sphere bundle of the resulting S^3

then

$x\in I^{\pm}(y)$ if the intersection number of Σ with Y is ±1

$x\in M\setminus J(y)$ if the intersection number is 0. □

Notes.

1) In dimensions other than 4, most of this carries straight across, except that for odd dimensions the nice relationship between the sign of the linking number and the information of which point is to the future of the other is lost.

2) The skies of M form a maximal family of S^2s in PN^I with the transitivity property that $\mathrm{Link}(X,Y)=\mathrm{Link}(Y,Z)=1 \Rightarrow \mathrm{Link}(X,Z)=1$.

3) If M is a conformally flat globally hyperbolic space-time with a

non-compact Cauchy surface, then $x \in I^+(y)$ if $\mathrm{Link}(X,Y)>0$, $x \in I^-(y)$ if $\mathrm{Link}(X,T)<0$, and $x \in M \setminus J(y)$ if $\mathrm{Link}(X,Y)=0$.

4) In curved space-times, the above results still hold locally in the sense that given any point $x \in M$ x has a causally convex globally hyperbolic neighbourhood with Cauchy slice $\mathbb{R}^3$, and for y lying in this set we obtain exactly the same relationship between linking of skies in the corresponding space of null geodesics as for Minkowski space.

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References.

[1] Greenberg and Harper, Algebraic topology: a first course.

[2] Rolfsen, Knots and Links.

Abstract

A twistor conformal field theory for four space-time dimensions

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A definition is proposed for four-dimensional conformal field theory in which a class of complex 3-manifolds and holomorphic sheaf cohomology replace the Riemann surfaces and holomorphic functions of two-dimensional theory. Suggestions are made about the interpretation of such a theory, in terms of the interaction of fundamental particles, in relation to the theory of twistor diagrams, and the possibility of extending it to incorporate gravity.

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