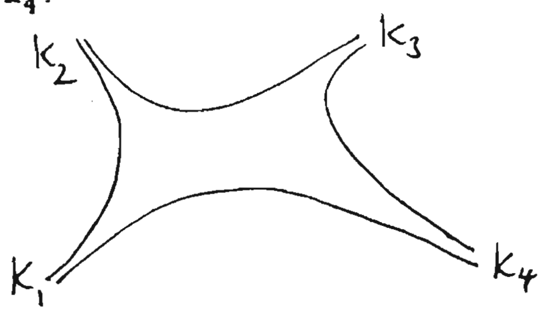
## Connections between amplitudes in string theory and twistor diagrams

In the proposal [1] for twistor conformal field theory in four dimensions (CFT4), an amplitude arises in the first instance as associated with a specific complex manifold $\mathbf{X}$. We then suggest that to obtain a conformally invariant amplitude there should be a summation of such amplitudes over (a class of) such complex manifolds. We hope to identify such conformally invariant amplitudes with the evaluations of twistor diagrams. Thus we are led to consider the possibility that at least some of the multiple integrations appearing in twistor diagrams are interpretable as integrations over manifold-defining parameters.

It is encouraging that the amplitudes calculated for the tree-diagrams of (bosonic) string theory can be put in the form of projective *spinor* integrals, interpretable in just this sense. This is only a matter of rewriting the calculations as given in Green, Schwarz and Witten, vol. 1, pp. 38-50, 355-390. As we shall see, a tantalising hint of an analogy with twistor diagrams then emerges.

The simplest example is that of integrating over the sheets formed by four open strings with spin 0, labelled by [26-dimensionall] momenta $k_1, k_2, k_3, k_4$.




$(k_1 + k_2 + k_3 + k_4 = 0)$

The argument is that first, such a sheet may be mapped conformally to a disc with four points on the boundary removed (equivalently, the upper-half-plane with four real points removed). Thus summation over all strings reduces to summing over conformally inequivalent discs (half-planes). This in turn reduces to a single integral over the cross-ratio of the four removed points, the range of integration being determined by the ordering (1234), well-defined up to cyclic permutation. This integral then produces the Veneziano beta-function appropriate to that ordering. Explicitly, using the upper-half-plane formulation, G. S. & W. write down

$$\int dz_1 dz_2 dz_3 dz_4 \; \exp(-k_1.k_2 \log|z_1 - z_2|) \exp(-k_1.k_3 \log|z_1 - z_3|)$$
$$\exp(-k_1.k_4 \log|z_1 - z_4|) \exp(-k_2.k_3 \log|z_2 - z_3|)$$
$$\exp(-k_2.k_4 \log|z_2 - z_4|) \exp(-k_3.k_4 \log|z_3 - z_4|)$$

where the integrand (i.e. the amplitude associated with a specific choice of $z_1$, $z_2$, $z_3$, $z_4$) is deduced from an action principle. They arrive at an integral over the cross-ratio by imposing $z_1 = 0$, $z_3 = 1$, $z_4 = \infty$, (thought of as "gauge-fixing") so that the integral reduces to

$$\int_0^1 dz_2 \, |z_2|^{-k_1.k_2} \, |1 - z_2|^{-k_2.k_3}$$

They have a argument to show that this is SL2C invariant iff the external momenta meet the "tachyon" conditions

$$-k_1^2 = k_1 . k_2 + k_1 . k_3 + k_1 . k_4 = 2 \quad \text{(in units of the Planck mass)}$$

and similarly for $k_2$, $k_3$, $k_4$.

Writing $s = (k_1 + k_2)^2 \qquad t = (k_1 + k_4)^2 \qquad u = (k_1 + k_3)^2$
these imply $s + t + u = -8$, $k_1.k_2 = (s + 4)/2$ etc.

However this invariance is (to me) more transparent and symmetrically expressed when the amplitude is written in terms of projective spinors The Green function $\log|z_1 - z_2|$ then appears as

$$\log \left| \frac{z_1 . z_2}{z_1 . \alpha \;\; z_2 . \alpha} \right|$$

where the spinor $\alpha$ corresponds to the point at infinity. SL2C invariance, i.e. independence of $\alpha$, is then obviously equivalent to the "tachyon" mass condition. If this is met, the integral then becomes

$$\int Dz_1 \dots Dz_4 \, [(z_1.z_2)(z_3.z_4)]^{-a} \, [(z_1.z_4)(z_2.z_3)]^{-b} \, [(z_1.z_3)(z_2.z_4)]^{-c}$$

where $a = k_1 . k_2 = 2 + s/2$; $\quad b = k_1 . k_4 = 2 + t/2$; $\quad c = k_1 . k_3 = 2 + u/2$,

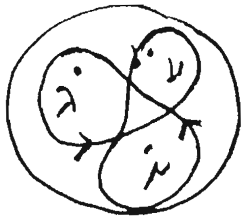
so $\qquad a + b + c = 2.$

and where the integration can be "freed" from the real line.

Such an integrand is familiar in spinor integral calculus. Note the symmetric contour integral formula

$$\frac{1}{(2\pi i)^2}\oint_C D\pi\,(\pi.\lambda)^{-a}(\pi.\mu)^{-b}(\pi.\nu)^{-c} = \frac{(\lambda.\mu)^{c-1}(\lambda.\nu)^{b-1}(\mu.\nu)^{a-1}}{\Gamma(a)\Gamma(b)\Gamma(c)}$$

$(a + b + c = 2)$

where C is the compact contour  in $\mathbf{CP}^1$



C is equivalent to a Pochhammer contour winding round a cut between two branch points, and so this compact contour integral formula implies the non-compact "beta-function" integral formula

$$\int_\lambda^\mu D\pi\,(\pi.\lambda)^{-a}(\pi.\mu)^{-b}(\pi.\nu)^{-c} = \frac{\pi}{\sin\pi a}\,\frac{\pi}{\sin\pi b}\oint_C (\pi.\lambda)^{-a}(\pi.\mu)^{-b}(\pi.\nu)^{-c}\,D\pi$$

$$= \frac{\Gamma(1-a)\,\Gamma(1-b)}{\Gamma(2-a-b)}\,(\lambda\mu)^{c-1}(\lambda.\mu)^{b-1}(\mu.\nu)^{a-1}$$

If we integrate out $z_2$ in this way (following G. S. and W.) we are left with

$$\frac{\Gamma(1-a)\Gamma(1-b)}{\Gamma(2-a-b)}\int Dz_1\,Dz_2\,Dz_3\;(z_1.z_3)^{-1}(z_3.z_4)^{-1}(z_4.z_1)^{-1}$$

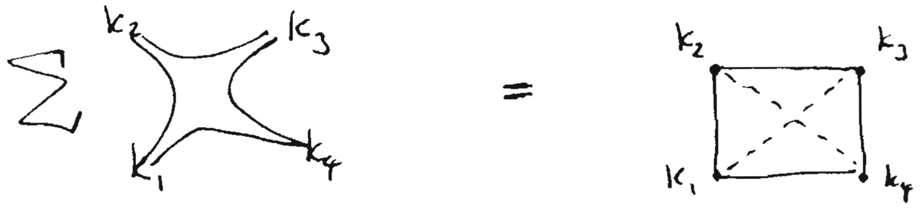
To G. S. and W. this remaining integral is a divergent integral which represents the volume of SL2R, and is to be divided out. As an alternative way of expressing this idea, we may change the contour to a compact $S^2 \times S^1$ and obtain the required finite result. Putting these ideas together, we may claim that the ***compact spinor contour integral***

$$\oint Dz_1\ldots Dz_4\;\frac{[(z_1.z_2)(z_3.z_4)]^{-k_1.k_2}}{\sin\pi(k_1.k_2)}\;\frac{[(z_1.z_4)(z_2.z_3)]^{-k_1.k_4}}{\sin\pi(k_1.k_4)}\;[(z_1.z_3)(z_2.z_4)]^{-k_1.k_3}$$

corresponds to an invariantly defined summation over strings, yielding the Veneziano amplitude

$$A(s,t) = \frac{\Gamma(-1-\tfrac{1}{2}s)\,\Gamma(-1-\tfrac{1}{2}t)}{\Gamma(-2-\tfrac{1}{2}s-\tfrac{1}{2}t)}$$

We might express this spinor integral by a "spinor diagram", so that

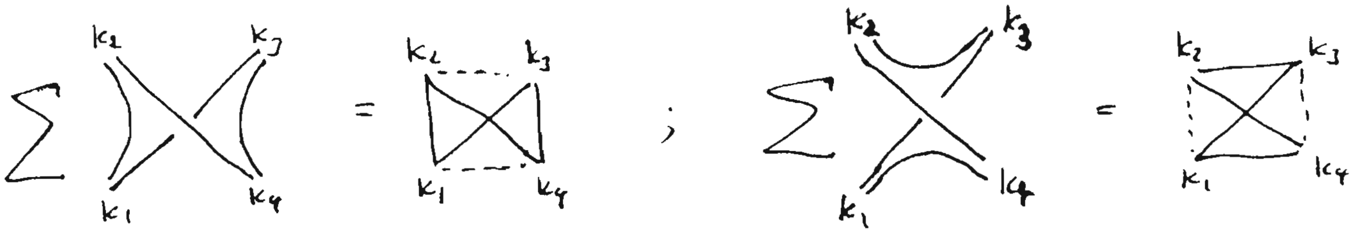



Here the solid lines represent factors $\dfrac{[(z_1.z_2)(z_3.z_4)]^{-k_1.k_2}}{\sin\pi(k_1.k_2)}$ , $\dfrac{[(z_1.z_4)(z_2.z_3)]^{-k_1.k_4}}{\sin\pi(k_1.k_4)}$

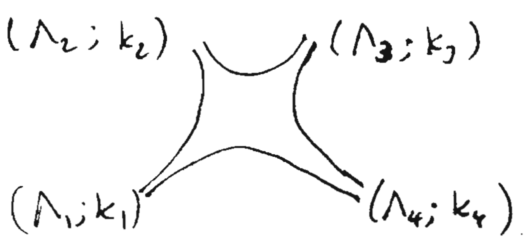
the dashed line a factor $[(z_1.z_3)(z_2.z_4)]^{-k_1.k_3}$ ( a *period* of the solid line)

and compact contour integration over the $z_i$ is implied.

By permuting the external states we have likewise

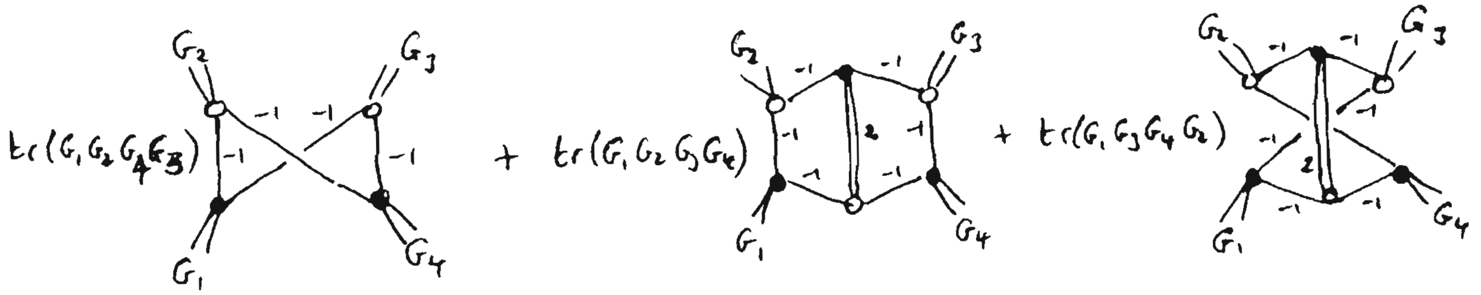



The significance of the ordering of the external states is seen in its connection with internal symmetry. Suppose the external states now also carry elements $\Lambda_i$ of U(n) as internal symmetry indices (such an element can be thought of as describing a quark-antiquark pair in the hadron model for which the original bosonic string theory was developed). Then according to string theory the amplitude for

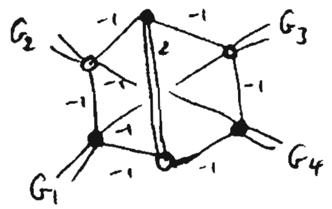



is assigned a coefficient of: $\mathrm{tr}(\Lambda_1\Lambda_2\Lambda_3\Lambda_4)$

Now it is very striking that such a trace structure, respecting order up to cyclic permutation, has already been observed in twistor diagram theory. As described in TN 23, the amplitude for pure SU(2) gauge field scattering takes the form




Note that these twistor integrals fall into a form analogous to the spinor integrals above, since the period of the logarithmic (-1)-line is just unity, i.e. each is a "period" of




Of course these integrals are entirely different from the spinor integrals above in that the external state parameters appearing in the exponents now correspond to *helicities* and not to *momenta*. Furthermore we have no action principle or Green function in the twistor picture to lend substance to these similarities.

But the analogy is close enough to give some more support to our conjecture that there exists an interpretation of these integrals as suitable invariant summations over twistor manifolds within a CFT4.

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[1] A.P.H., R.P. and M.A.S., to appear in *Physics Letters*, see also TN 26.