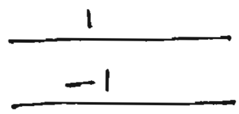
Notes on the correspondence between Feynman diagrams and twistor diagrams

In **TN25** we established a remarkable twistor-diagrammatic formula for second-order massless ϕ^4 scattering and deduced (a) some extensions to higher-order ϕ^4 amplitudes and (b) a new viewpoint on the crossing symmetry problem for the first-order ϕ^4 amplitude. Here are some notes on the features that arise on attempting to develop these ideas more generally and systematically as a "translation" procedure.

§1. One clear feature emerging from these ϕ^4 diagrams is the pair of lines

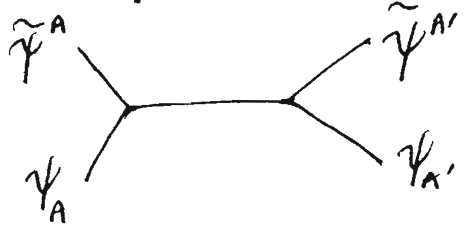


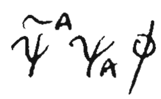
representing the "off-shell" scalar propagator $\Delta_F(x - y)$

with the "on-shell" propagator 

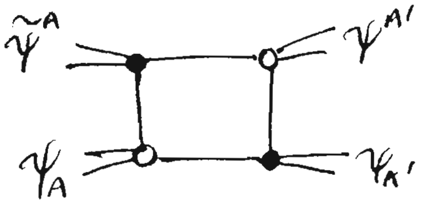
as its "period".

Note that this scalar propagator can be exhibited in a simpler context, namely the first-order process



where the vertex is given by the so-called "Yukawa interaction" of massless fields expressed by the Lagrangian 

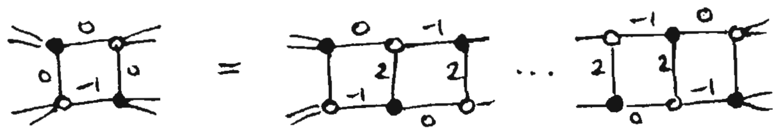
Standard arguments translate this first-order amplitude into

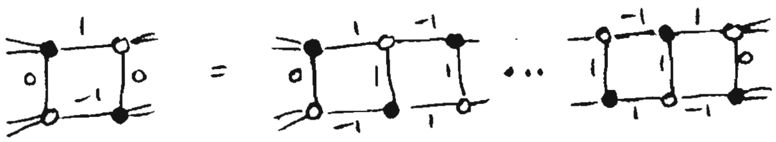


One can think of the massless spin-1/2 fields as *test-functions* for the twistor representation of $\Delta_F(x - y)$. It is generally useful and important to include the conformally invariant "Yukawa" interaction along with the ϕ^4 interaction in studying the structure of higher order diagrams.

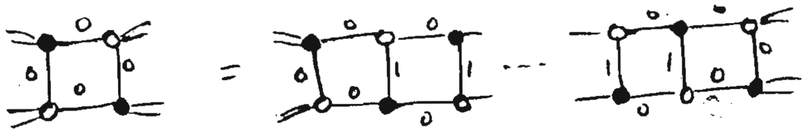
§2. The scalar propagator then fits into a scheme together with

for massless spin-1/2, spin-1 propagators respectively. In each case the propagator can be regarded as an operator which projects out an eigenstate of spin, in such a way that the operator is idempotent. Thus:





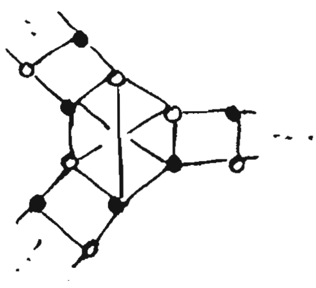
The contours here are consistent with



If Feynman propagators Δ_F are equivalent to these "chains" then how could Feynman *vertices*, of essential form

$$\int \Delta_F(x-x_1)\ \Delta_F(x-x_2)\ \Delta_F(x-x_3)\ d^4x\,,$$

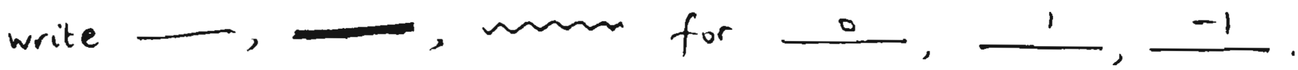
be translated? One might expect twistor-diagram vertices of form:



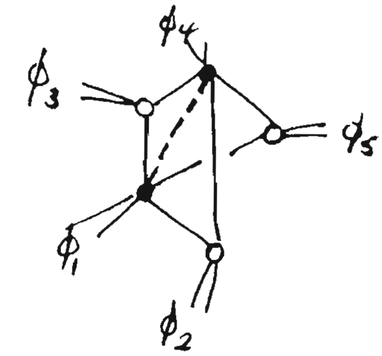
For this to be true, a necessary condition is that we can differentiate w.r.t. x_1, x_2 and x_3, thus reducing each Δ_F to the δ-function, and get the right answer - as evaluated by using six massless fields [two in x_1, two in x_2, two in x_3] as "test-functions". The form of the derivatives will depend upon which particular helicities we are looking at, but essentially the problem boils down to establishing a twistor-diagrammatic formula, taking this form, for the scalar ϕ^6 integral

$$\int \phi_1(x)\,\phi_2(x)\,\phi_3(x)\,\phi_4(x)\,\phi_5(x)\,\phi_6(x)\ d^4x$$

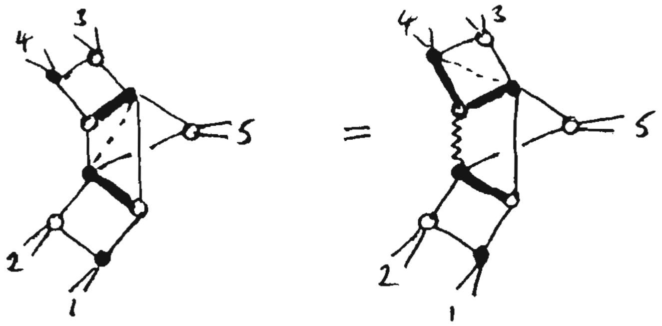
§3. There *is* a naturally suggested structure of this form for the ϕ^6 integral. It's useful here to use an abbreviated twistor diagram notation:



Now let us begin with ϕ^5. Certainly the diagram

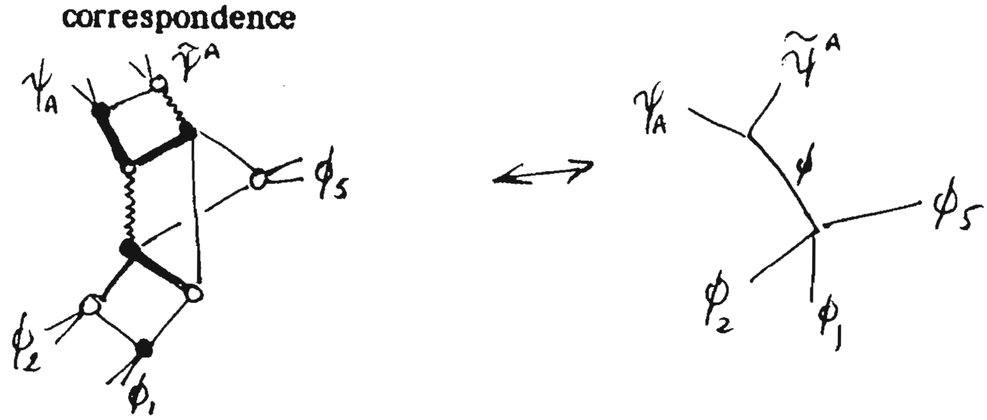


represents the ϕ^5 integral in one channel (235-14). The guess is that

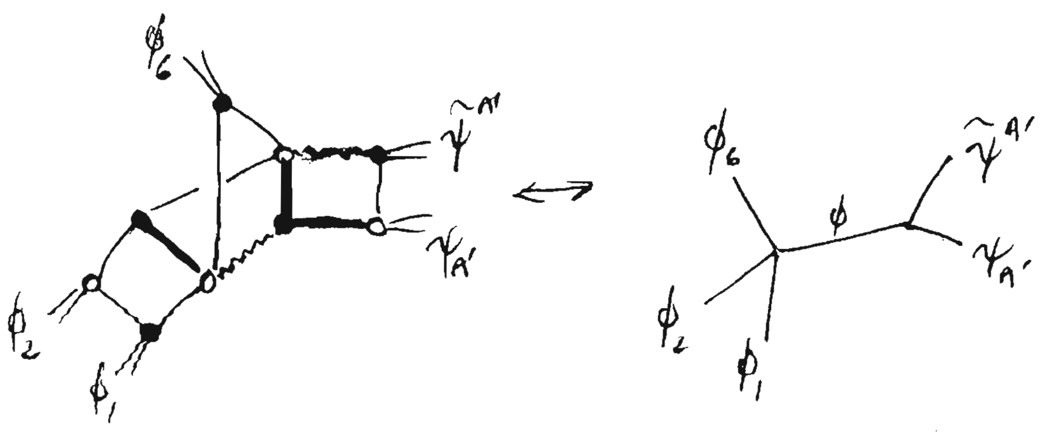


(using integration by parts)

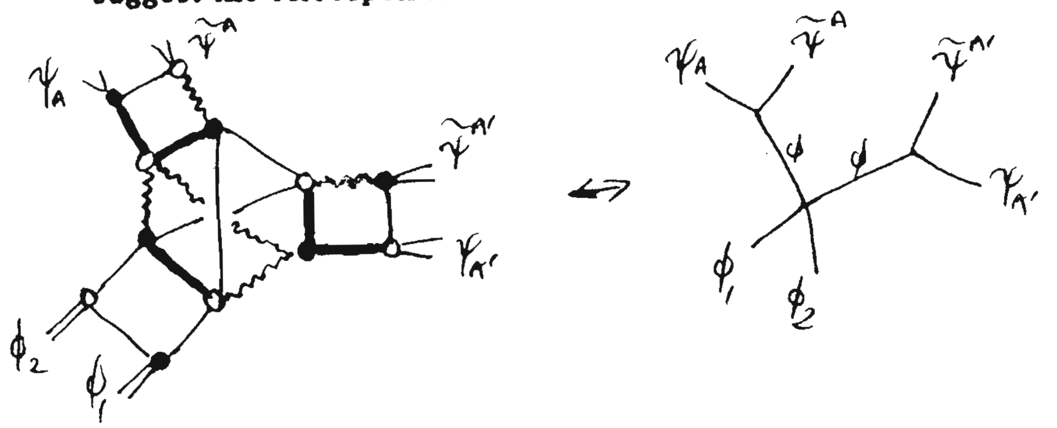
is valid in all channels. This would imply the (conformally invariant) correspondence



(again using simple integration by parts). If so we have also

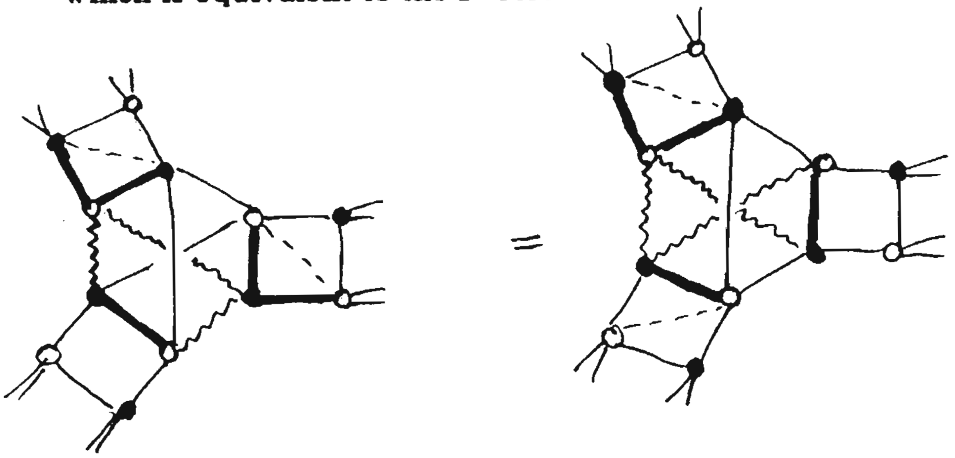


By "superimposing" these (i.e. looking at period structure) we are led to suggest the correspondence



(*)

which is equivalent to the ϕ^6 formulas



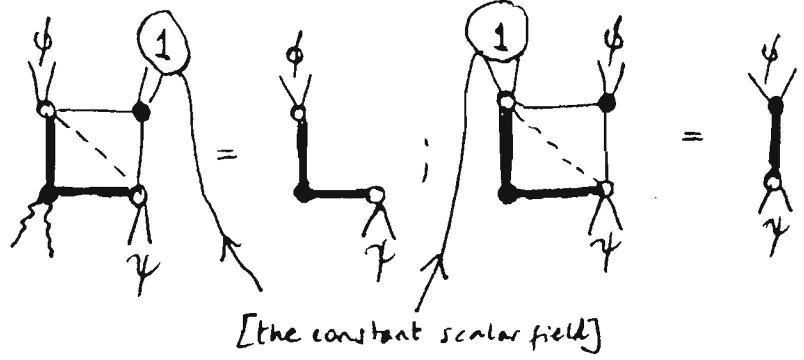
etc. etc.

(by permutation symmetry). This is in turn is the key to describing all Feynman vertices in twistorial terms, as indicated in §1.

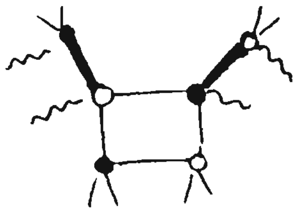
These are only guesses, but they embody important constraints on what formulas could possibly be correct. For any possible ϕ^6 formula implies a formula for the Feynman diagram (*), which one would expect (at least naively; see below) to be conformally invariant, and to have periods related to known integrals. What we have done above is to construct formulas which have a chance of satisfying these strong conditions. Unfortunately they have so far proved too difficult to check explicitly.

Another highly significant constraining feature of these integrals arises from the limiting cases in which one external massless field is allowed to tend towards the *constant* field. In the twistor picture this is a geometrical limit arising as the point defining an elementary state moves towards I. In the space-time integrals the effect is that of reducing ϕ^6 to ϕ^5 or ϕ^5 to ϕ^4. These limits must all make sense and agree - a strong condition.

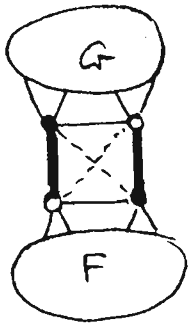
In particular we require contours which justify:



with the numerator factor involving $I^{\alpha\beta}$ "cancelling" the effect of the external parameters tending to $I^{\alpha\beta}$. At first sight, this is impossible: integration by parts seems to show that the limit has to be zero. But it *does* seem to be possible if the external fields are attached in the way suggested in **TN25**, viz. as



This problem is connected with the puzzle noticed long ago by RP, namely that one may formally deduce



for the inner product of two 2-twistor functions of mass zero, and yet simple integration by parts shows that such an integral must vanish.

A further question concerns the assumption made above that higher-order diagrams built out of conformally invariant interactions, should themselves be conformally invariant functionals. In standard QFT conformal symmetry breaking is necessary for renormalisation and so it is possible that in the twistor picture also conformal symmetry breaking has to play some essential role in building up diagrams.

Putting these points together, it looks very likely that the formulae suggested above cannot actually make sense without some modification of the geometry of twistor space at **I** - something expected anyway for the description of mass (and gravity). My hope is that when correctly interpreted, the twistor diagrams written down above will form the basis of a systematic calculus.

A. P. Hodges