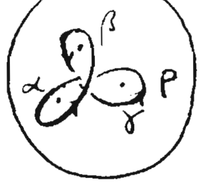
A note on Pochhammer contours

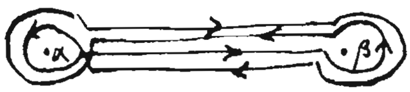
The formula

$$\oint_P \frac{D\pi}{(\pi.\alpha)^a(\pi.\beta)^b(\pi.\gamma)^c} = \frac{(2\pi i)^2(\alpha.\beta)^{c-1}(\beta.\gamma)^{a-1}(\gamma.\alpha)^{b-1}}{\Gamma(a)\,\Gamma(b)\,\Gamma(c)}$$



(a + b + c = 2)

can be demonstrated by deforming the symmetrical contour P to



Provided Re(b) and Re(c) are both > -1, this integral is:

$(1 - e^{2\pi ia})(1 - e^{2\pi ib})$ x (standard noncompact beta-function integral)

+ (contributions from small arcs vanishing as radii tend to 0)

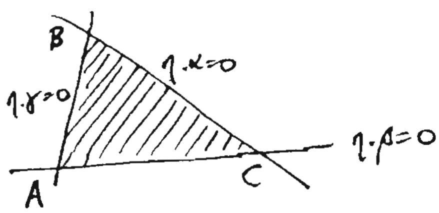
Extension to all non-integral a, b, c then follows by analytic continuation.

This formula and its justification generalises to $\mathbf{CP}^2$ (and presumably to $\mathbf{CP}^n$); namely we have for a 2-dimensional Pochhammer contour P,

$$\oint_P \frac{D^2\eta}{(\eta.\alpha)^a(\eta.\beta)^b(\eta.\gamma)^c(\eta.\delta)^d} = \frac{(2\pi i)^3(\alpha\beta\gamma)^{d-1}(\beta\gamma\delta)^{a-1}(\gamma\delta\alpha)^{b-1}(\delta\alpha\beta)^{c-1}}{\Gamma(a)\,\Gamma(b)\,\Gamma(c)\,\Gamma(d)}$$

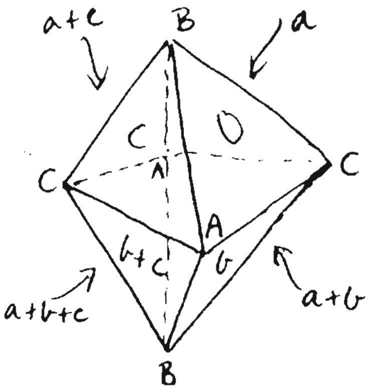
(a + b + c + d = 3)

In this case, to define the contour P, consider the noncompact integral of the given form over a triangular region:



The integral can be evaluated explicitly. Choose coordinates u, v for $\mathbf{CP}^2 - \{\eta\cdot\alpha=0, \eta\cdot\beta=0, \eta\cdot\gamma=0, \eta\cdot\delta=0\}$ such that $u = \eta\cdot\alpha/\eta\cdot\delta$, $v = \eta\cdot\beta/\eta\cdot\delta$.
The resulting integral, a standard extension of the beta function integral, is finite provided Re(a), Re(b), Re(c) > -1.

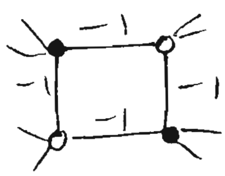
Now, we need to glue together eight copies of this triangle, together with sections with radius ε round the branch points.
One may check that the triangles fit togther like the faces of an octahedron, yielding a closed contour of topology S^2, and yield the integral as stated.



Triangular face labelled "b" indicates that the triangle is on the sheet characterised by factor $e^{2\pi ib}$ (etc.)

Special cases when one (or more) of the complex exponents is actually an integer. In the original one-dimensional case, the Pochhammer contour of topology S^1 becomes equivalent to the sum of two disjoint S^1 's (i.e. to $S^0 \times S^1$).
In the two dimensional integral what happens is that one "vertex" of the octahedron can be identified with the opposite vertex, so that the S^2 can be deformed into a torus $S^1 \times S^1$. Or this can be seen directly from the form of the integral: if d is an integer there is clearly a contour constructed as (small circle round $\eta \cdot \delta = 0$) x (one-dimensional Pochhammer contour inside the $\mathbf{CP}^1$ ($\eta \cdot \delta = 0$)).

Higher-dimensional Pochhammer contours of this kind play an essential role in evaluating such twistor diagrams as



where many logarithmic factors are to be convoluted.

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