A note on Pochhammer contours

The formula

\[
\oint \frac{D\eta}{(\eta.x)^a(\eta.y)^b(\eta.z)^c} = \frac{(2\pi)^2 (\alpha, \beta)_{\xi}^{-1} (\beta, \gamma)_{\zeta}^{-1} (\gamma, \omega)_{\eta}^{-1}}{\Gamma(\alpha) \Gamma(\beta) \Gamma(\gamma) \Gamma(\omega)}
\]

\((a + b + c = 2)\)

can be demonstrated by deforming the symmetrical contour \(P\) to

Provided \(\text{Re}(b)\) and \(\text{Re}(c)\) are both \(> -1\), this integral is:

\((1 - e^{2\pi ia})(1 - e^{2\pi ib})\pi\) (standard noncompact beta-function integral)

+ (contributions from small arcs vanishing as radii tend to 0)

Extension to all non-integral \(a, b, c\) then follows by analytic continuation.

This formula and its justification generalises to \(\mathbb{CP}^2\) (and presumably to \(\mathbb{CP}^n\)); namely we have for a 2-dimensional Pochhammer contour \(P\),

\[
\oint \frac{D\eta}{(\eta.x)^a(\eta.y)^b(\eta.z)^c(\eta.z)^d} = \frac{(2\pi)^2 (\alpha, \beta)_{\xi}^{-1} (\beta, \gamma)_{\zeta}^{-1} (\gamma, \omega)_{\eta}^{-1} (\omega, \delta)_{\eta}^{-1}}{\Gamma(\alpha) \Gamma(\beta) \Gamma(\gamma) \Gamma(\omega) \Gamma(\delta)}
\]

\((a + b + c + d = 3)\)

In this case, to define the contour \(P\), consider the noncompact integral of the given form over a triangular region:

![Diagram of a triangular region]

The integral can be evaluated explicitly. Choose coordinates \(u, v\) for

\(\mathbb{CP}^2\) - \((q.x=0, q.y=0, q.z=0)\) such that \(u = q.x/q.z, v = q.y/q.z\)

The resulting integral, a standard extension of the beta function integral, is finite provided \(\text{Re}(a), \text{Re}(b), \text{Re}(c) > -1\).
Now, we need to glue together eight copies of this triangle, together with sections with radius $\varepsilon$ round the branch points.
One may check that the triangles fit together like the faces of an octahedron, yielding a closed contour of topology $S^2$, and yield the integral as stated.

![Diagram of an octahedron with labels and arrows indicating the gluing process.]

Triangular face labelled "b" indicates that the triangle is on the sheet characterised by factor $e^{2\pi ib}$ (etc.)

Special cases when one (or more) of the complex exponents is actually an integer. In the original one-dimensional case, the Pochhammer contour of topology $S^1$ becomes equivalent to the sum of two disjoint $S^1$'s (i.e. to $S^0 \times S^1$).

In the two dimensional integral what happens is that one "vertex" of the octahedron can be identified with the opposite vertex, so that the $S^2$ can be deformed into a torus $S^1 \times S^1$. Or this can be seen directly from the form of the integral: if $d$ is an integer there is clearly a contour constructed as (small circle round $\gamma \cdot \delta = 0$) $\times$ (one-dimensional Pochhammer contour inside the CP$^1$ $(\gamma \cdot \delta = 0$)).

Higher-dimensional Pochhammer contours of this kind play an essential role in evaluating such twistor diagrams as

![Diagram of a twistor diagram with nodes and arrows indicating the flow of fields.]

where many logarithmic factors are to be convoluted.

Andrew Hodges