Cohomology Reduction through Qadir's Intertwining Operators

In his article on "Qadir's intertwining operators" (TN 25), R.P. described certain twistor contour integrals first suggested by A. Qadir in his Ph.D. thesis (London, 1970). The original use of these operators was to pass from representations of $SU(2,2)$ etc. to equivalent ones. Here, I wish to describe their cohomology reducing properties.

Qadir's integrals

Qadir's integrals (short QI) deal with holomorphic functions in four twistor variables W, X, Y, Z, homogeneous of degrees w, x, y, z respectively. Additionaly, these functions must satisfy

$$(1) \quad f(W+\lambda X+\mu Y+\gamma Z, X+\rho Y+\sigma Z, Y+\tau Z, Z) = f(W,X,Y,Z) \qquad (\forall\, \lambda,\mu,\gamma,\rho,\sigma,\tau).$$

In their original form, QIs are formal contour integrals

$$(2) \quad \begin{aligned} a)\quad & g_1(W,X,Y,Z) = \oint f(W,X,Z,Y+\lambda Z)\, d\lambda \\ b)\quad & g_2(W,X,Y,Z) = \oint f(W,Y,X+\lambda Y,Z)\, d\lambda \\ c)\quad & g_3(W,X,Y,Z) = \oint f(X,W+\lambda X,Y,Z)\, d\lambda \end{aligned}$$

and some higher versions. Each expression satisfies (1) and is homogeneous of degrees

$$g_1 : (w, x, z+1, y-1) \qquad g_2 : (w, y+1, x-1, z) \qquad g_3 : (x+1, w-1, y, z).$$

We restrict ourselves to the case that w, x, y, z are integers. Condition (1) and homogeneity imply that f can be rewritten as a function of higher valence skew simple twistors:

$$F(\overline{WXYZ}, \overline{XYZ}, \overline{YZ}, \overline{Z}) := f(\overline{W}, \overline{X}, \overline{Y}, \overline{Z})$$

F is also homogeneous, of degrees $(w, x-w, y-x, z-y)$ in the respective variables. Hence, F actually represents a section of some line bundle over F_{123}, the flag manifold of projective twistor space (points $\subset$ lines $\subset$ planes $\subset$ $\mathbf{PT}$). The dependence of F on the first variable is in a sense trivial; indeed, the value of the first homogeneity degree does not alter the line bundle F represents. Therefore, the first variable is ignored in what follows.

We denote the sheaf of sections of this line bundle by $\mathcal{O}_{(x-w,z-y)[y-x]}$. Abstractly,

$$\mathcal{O}_{(p,r)[q]} := \eta_3^*\mathcal{O}(p) \otimes \eta_1^*\mathcal{O}(r) \otimes \eta_2^*\mathcal{O}[q]$$

where η_i^* are pullback maps for the fibrations $\eta_i : \mathsf{F}_{123} \to \mathsf{F}_i$.
($\mathsf{F}_1 = \mathbf{PT}$, $\mathsf{F}_2 = \mathbf{M}$, $\mathsf{F}_3 = \mathbf{PT}^*$).

Cohomology reduction

The aim of this article is to show that there are isomorphisms

$$
(3)\qquad
\begin{aligned}
&a)\quad H^n(\mathsf{F}_{123}, \mathcal{O}(p,-r-2)[q]) \xrightarrow{\simeq} H^{n-1}(\mathsf{F}_{123}, \mathcal{O}(p,r)[q-r-1]) && (r \geqslant 0)\\
&b)\quad H^n(\mathsf{F}_{123}, \mathcal{O}(-p-2,r)[q]) \xrightarrow{\simeq} H^{n-1}(\mathsf{F}_{123}, \mathcal{O}(p,r)[q-p-1]) && (p \geqslant 0)\\
&c)\quad H^n(\mathsf{F}_{123}, \mathcal{O}(p,r)[-q-2]) \xrightarrow{\simeq} H^{n-1}(\mathsf{F}_{123}, \mathcal{O}(p-q-1,r-q-1)[q]) && (q \geqslant 0)
\end{aligned}
$$

which are realised by the QIs a), b) and c) respectively.

The most interesting of these is case b); the others behave similarly. The proof of b) involves direct images along the fibration

$$\mu : \mathsf{F}_{123} \longrightarrow \mathsf{F}_{13} = \mathsf{A}.$$

We require an interesting non-standard vector bundle over A, which can best be described by using a third spinor-type index.

DEFINITION . *The bundle $S_{\hat{A}}$ on A is given by its fibre at $[W_\alpha, Z^\beta] \in \mathsf{A}$*

$$S_{\hat{A}}\big|_{[W_\alpha, Z^\beta]} := \{\textit{simple skew } R^{\alpha\beta} \textit{ such that } W_\alpha R^{\alpha\beta} = 0 = R_{\alpha\beta} Z^\beta\}.$$

As for the usual spinors, there is a skew product on $S_{\hat{A}}$:

$$S_{\hat{A}} \wedge S_{\hat{B}} \cong S(-1,-1)$$

This is given explicitly by $(P^{\alpha\beta}) \wedge (R^{\gamma\delta}) \mapsto P^{\alpha\beta} R_{\beta\gamma}$ on the fibre over $[W_\alpha, Z^\beta]$. To see that this gives the desired result, write $P^{\alpha\beta} = Z^{[\alpha} X^{\beta]}$ and $R_{\beta\gamma} = Y_{[\beta} W_{\gamma]}$. X and Y must satisfy the incidence relations $Z^\alpha Y_\alpha = 0 = W_\alpha X^\alpha$, and therefore

$$P^{\alpha\beta} R_{\beta\gamma} = \tfrac{1}{4} Z^\alpha W_\gamma X^\beta Y_\beta \in \langle W_\gamma \rangle \otimes \langle Z^\alpha \rangle = S(-1,-1)\big|_{[W_\gamma, Z^\alpha]}$$

as required. As usual, the sheaf of sections of $S_{\hat{A}}$ is denoted by $\mathcal{O}_{\hat{A}}$.

We also need some vanishing statements (similar to the ones given in [EPW]).

VANISHING LEMMA. *For $\mu : \mathsf{F}_{123} \to \mathsf{F}_{13}$,*

$$\mu^i_* \mathcal{O}_{\hat{A}\ldots\hat{B}}(k,l)[m] = 0 \qquad \begin{cases} \forall\, i \neq 0 & \textit{if } m \geqslant 0 \\ \forall\, i & \textit{if } m = -1 \\ \forall\, i \neq 1 & \textit{if } m \leqslant -2 \end{cases}$$

Fibres of μ are $\mathbf{CP}_1$, and direct images give the fibrewise cohomology, and so the lemma follows from the usual vanishing theorems for cohomology of homogeneous sheaves on $\mathbf{CP}_1$.

The proof of b) starts with the short exact sequence

$$\mathsf{F}_{123}: \quad 0 \longrightarrow \mathcal{O}(p,r)[-q-2] \xrightarrow{\pi_{\hat{A}}\cdots\pi_{\hat{B}}} \mathcal{O}_{\underbrace{(\hat{A}\ldots\hat{B}\hat{C})}_{q+1}}(p,r)[-1] \xrightarrow{\pi^{\hat{C}}} \mathcal{O}_{\underbrace{(\hat{A}\ldots\hat{B})}_{q}}(p-1,r-1)[0] \longrightarrow 0$$

Multiplication by $\pi_{\hat{A}}$ is interpreted as the natural inclusion $\mathcal{O}[-1] \hookrightarrow \mathcal{O}_{\hat{A}}$ (on F_{123}), and $\pi^{\hat{C}} = \epsilon^{\hat{C}\hat{D}}\pi_{\hat{D}}$ requires the symplectic form $\epsilon^{\hat{C}\hat{D}} \in \mathcal{O}^{[\hat{C}\hat{D}]}(-1,-1)$. Taking cohomology along fibres of μ gives a long exact sequence of direct image sheaves. Because of the vanishing lemma, the only nonzero part of this long exact sequence is the coboundary map

$$\mu_*\mathcal{O}_{(\hat{A}\ldots\hat{B})}(p-1,r-1)[0] \xrightarrow{\simeq} \mu_*^1\mathcal{O}(p,r)[-q-2].$$

Now we can use

$$\begin{aligned} \mathsf{F}_{13}: \quad & \mathcal{O}^{(\hat{A}\ldots\hat{B})} \cong \mu_*\mathcal{O}[q] \\ & \Longrightarrow \mathcal{O}_{(\hat{A}\ldots\hat{B})} \cong \mu_*\mathcal{O}(-q,-q)[q] \end{aligned}$$

to get

$$\mu_*^1\mathcal{O}(p,r)[-q-2] \cong \mu_*\mathcal{O}(p-1-q,r-1-q)[q]. \tag{4}$$

Finally, the move back to F_{123} is accomplished with the Leray spectral sequence:

$$E_2^{p,q} = H^q(\mathsf{F}_{13}, \mu_*^p\mathcal{F}) \Longrightarrow H^{p+q}(\mathsf{F}_{123}, \mathcal{F}) \qquad \mathcal{F} \text{ sheaf on } \mathsf{F}_{123}.$$

For both sheaves involved in (4), the corresponding spectral sequences are degenerate (again by the vanishing lemma). Hence

$$\begin{aligned} H^n(\mathsf{F}_{123}, \mathcal{O}(p,r)[-q-2]) &\cong H^{n-1}(\mathsf{F}_{13}, \mu_*^1\mathcal{O}(p,r)[-q-2]) \\ &\cong H^{n-1}(\mathsf{F}_{13}, \mu_*\mathcal{O}(p-q-1,r-q-1)[q]) \\ &\cong H^{n-1}(\mathsf{F}_{123}, \mathcal{O}(p-q-1,r-q-1)[q]). \end{aligned} \tag{5}$$

as desired. □

Starting with (3), one can go on to prove the Borel–Weil theorem on F_{123} by stringing isomorphisms together.

Now, where do QIs come in? Again, take case b) in (2) as an example:

$$G(\overset{+++}{XYZ}, \overset{++}{YZ}, \overset{|}{Z}) = \oint F(\overset{+++}{XYZ}, \overset{++}{XZ} + \lambda\overset{++}{YZ}, \overset{|}{Z})\, d\lambda \tag{6}$$

in F_{123}-compatible notation. The integration takes place over branched contours on the fibres of $\mu : \mathsf{F}_{123} \to \mathsf{F}_{13}$ (the point $\lambda = \infty$ must be included), and the homogeneity changes from $(p, -q-2, r)$ in F to $(p-q-1, q, r-q-1)$ in G.

Given a Stein neighbourhood U in F_{13}. Then, (6) maps first Čech cohomology representatives on $\mu^{-1}(U)$ to global functions on $\mu^{-1}(U)$ and preserves cohomology classes. In other words, we have a map

$$\oint : \quad \mu^1_* \mathcal{O}_{(p,r)[-q-2]}(U) \xrightarrow{\simeq} \mu_* \mathcal{O}_{(p-1-q,r-1-q)[q]}(U). \tag{7}$$

This involves picking a (suitably nice) Stein cover of $\mu^{-1}(U)$, and choosing a (smooth) family of branched contours (one on each fibre over U) compatible with this cover. Roughly, this means that each double overlap of the cover (restricted to a fibre) contains exactly one branch. (Branched contours are explained in detail in "Spinors and Spacetime II"). Then, a splitting process due to Spaling and Ward (See **TN** 1) is performed to give the result.

All this is just a generalisation of the usual zero rest mass integral procedure.

Anyway, by choosing the U's to be the n-fold overlaps of a (nice) Stein cover of F_{13}, (7) induces the map

$$\oint : \quad H^{n-1}(\mathsf{F}_{13}, \mu^1_* \mathcal{O}_{(p,r)[-q-2]}) \xrightarrow{\simeq} H^{n-1}(\mathsf{F}_{13}, \mu_* \mathcal{O}_{(p-q-1,r-q-1)[q]})$$

which realizes the middle step of (5).

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Coming soon . . . .

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