NON-HAUSDORFF TWISTOR SPACES FOR KERR AND SCHWARZSCHILD

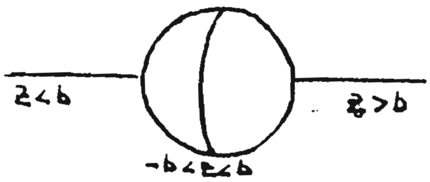
It is well-known that Einstein's equations for stationary, axi-symmetric vacuum space-times can be reduced to a form of the rank 2 anti-self-dual Yang-Mills equations by the introduction of Weyl co-ordinates (see, for example, Witten 1979). N.M.J.W. and L.J.M. (1988) showed that if these are solved locally by using the usual Ward transform, the holomorphic vector bundle over ordinary twistor space is actually the pull-back of a bundle E over a non-Hausdorff 'reduced' space.

Weyl co-ordinates are notorious for concealing the interesting parts of space-time geometry. For example, in the Schwarzschild solution, they only represent $R \geqslant 2m$, and the horizon is a part of the symmetry axis. (I shall use R to denote the radial co-ordinate in Kerr and Schwarz-schild.) Since all the information about the analytic continuation of the manifold is contained in the exterior part of the metric, one might expect to find it in the twistor description; and this is in fact possible. The Kerr and Schwarzschild solutions have reduced twistor spaces which consist of two Riemann spheres S_0 and S_1 which are identified except at three pairs of points; these points are the points at infinity and $w = \pm(m^2-a^2)^{\frac{1}{2}}$ where w is a co-ordinate on the spheres and $a = 0$ in the Schwarzschild case. The bundle can be described in the standard form which consists of first restricting it to each sphere and then giving the patching matrix P between them. In each case, $E|_{S_0}$ is $L_1 \oplus L_0$; $E|_{S_1}$ is $L_{-1} \oplus L_0$; and

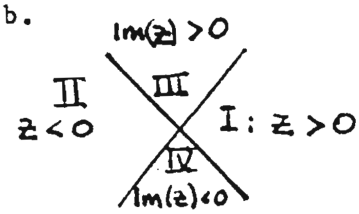
$$P = \frac{1}{w^2-m^2+a^2}\begin{pmatrix} (w+m)^2 + a^2 & 2am \\ 2am & (w-m)^2 + a^2 \end{pmatrix} .$$

(again, $a = 0$ for Schwarzschild). This description is unique if we demand that P be real (in the sense $\overline{P(w)} = P(\bar{w})$) and symmetric, and provided we know which points belong to which sphere in the reduced twistor space. We can, however, obtain a different patching matrix from the same bundle by changing our minds about which of the double points belongs to each of S_0 and S_1, and then putting the bundle in standard form over the two new spheres.

We therefore have four different possibilities. If we take S_0 and S_1 to be labelled by the points at infinity, then as well as the original description we can swap the points at +b, at −b or at both, where $b = (m^2-a^2)^{\frac{1}{2}}$. In order to see what this means in terms of the space-time, we have to introduce the idea of the patching matrix's being 'adapted' to one part of the axis in the (z,r)-plane. (Here z and r are the co-ordinates on the space of the orbits of the Killing vectors in the space-time; r=0 represents the symmetry axis.) This simply means that we can find the metric on the space of Killing vectors on this part of the axis by taking the limit as $r \to 0$ of its value in a neighbourhood of it. The patching matrix P given above is adapted to $z > b$, which corresponds to one half of the axis of symmetry in the space-time, outside the horizon. If we interchange the points at w=b, we get a matrix adapted to $-b<z<b$, which is the (outer) horizon; and if we interchange the points at both w=b and w=−b, we have a patching matrix adapted to the other half of the axis, where $z<-b$.



In each case, for both Kerr and Schwarzschild, the bundle E restricted to S_0 is $L_1 \oplus L_0$ and the metric on the space of Killing vectors can be extended analytically to the axis or horizon. Moreover, in the region where $-b<z<b$, we can continue this metric to the region where it is negative definite. To do this, we use the same construction as before (see Woodhouse and Mason 1988), but take values of r which are purely imaginary. Thus an orbit of the Killing vectors, which is representd by the pair of points $w = z + ir$ and $w = z - ir$, now corresponds to a pair of points on the real axis in the reduced twistor space. By taking r to be both positive and negative (when it is real) and Im(r) to be both positive and negative (when r is imaginary), we can construct the usual cross-over at $R = m + b$.



We can now choose either to identify regions I and II, and regions III and IV, or to put in the orbit (z,r) = (b,0) which corresponds to the cross-over itself. It can be shown that regularity of the metric (on the space of Killing vectors) at this point depends on the singularity structure of the patching matrix at $w=b$.

If we swap the points $w=-b$ in the Kerr solution, we find that we get another patching matrix Q of the kind that produces a cross-over (that is, Q is negative definite on the real axis in the w-plane); and that E restricted to S_0 is still $L_1 \oplus L_0$. If we take Q to be adapted to $-b<z<b$, then we can construct a similar picture to the one above; and the patching matrix adapted to $z>b$ turns out to be the inverse of the original matrix P. Since this can be obtained from P by replacing m with $-m$, the exterior region is now a negative mass Kerr solution. This of course must contain the ring singularity; the conjecture is that this is represented by the pull-back of E to the fibre of (Euclidean) twistor space above the appropriate points being non-trivial.

To obtain the Penrose diagram for the Kerr solution (see, for example, Hawking and Ellis 1973 p165) we have to identify region III for the '+b' crossover with region IV for the '-b' crossover. This can be done by considering the effect on the patching matrices of a reflexion of the (z,r) plane in the line $z = 0$. We also have to do the conformal rescaling which allows us to adjoin $\mathscr{I}$ to the space-time; it is at the moment less clear how the possibility of doing this is shown up by the twistor picture.

What is clear, however, is the difference between the Kerr and Schwarzschild solutions. For the latter, interchanging the points at $w=+m$ gives the same cross-over picture as for Kerr; but when we make the switch at $w=-m$, we find that the bundle E restricted to the new S_0 becomes $L_2 \oplus L_{-1}$. It is straightforward to see that this leads to a pole in the metric on the space of Killing vectors as $r \to 0$; this is of course the usual curvature singularity at $R = 0$.

References

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