A twistor transform for Hermitean symmetric spaces.

A familiar construction in twistor theory is the twistor transform which expresses the fact that a zero-rest-mass field can be given by a contour integral of a function of homogeneity $-n-2$ on twistor space $\mathbb{PT}^+$ or $+n-2$ on dual twistor space $\mathbb{PT}^{*-}$. Coupled with the conjugation $Z^\alpha \mapsto \bar{Z}_\alpha$ from $\mathbb{PT}^- \to \mathbb{PT}^{*-}$ and the pairing (called the dot product)

$$H^1(\overline{\mathbb{PT}^+}, \mathcal{O}(-n-2)) \times H^1(\overline{\mathbb{PT}^-}, \mathcal{O}(n+2)) \xrightarrow{\cdot} H^3(\mathbb{PT}, \Omega^3) \cong \mathbb{C}$$

(which is cup product to $\overline{\mathbb{PT}^+} \cap \overline{\mathbb{PT}^-} = \mathbb{PN}$ followed by Meyer-Vietoris connecting homomorphism) this gives a construction of certain unitary representations of $SU(2,2)$. The great advantage of this construction is that it is inherently geometrical (which allows one, for example, to define elementary states & calculate them using relative cohomology). In (1,2) MGE & I have extended twistor theory as an $SU(2,2)$ construction to arbitrary $\mathbb{C}$-semi-simple Lie groups. This note is a brief summary of what is known about possible generalizations of the twistor transform to other groups & spaces.

<u>An example</u>: In the usual twistor theory we use the Hermitean form $Z^\alpha \bar{Z}_\alpha$; consider instead $Sp(2,\mathbb{C})$ acting on $\mathbb{C}^4$ & preserving a symplectic form ω. In place of $SU(2,2)$ use $Sp(2,\mathbb{R})$. Then conjugation is $Z^\alpha \longrightarrow \bar{Z}^\alpha$ (conj. componentwise) & instead of $\mathbb{PT}^\pm$ etc we can consider $(\mathbb{CP}^3)^\pm = \{[V] \in \mathbb{CP}^3 \mid i\omega(V,\bar{V}) \gtrless 0\}$. The machine (Penrose Transform) of [1], [2] shows that

$$n \geq 0: \quad H^1(\overline{(\mathbb{CP}^3)^+}, \mathcal{O}(-3-n)) \cong H^1(\overline{(\mathbb{CP}^3)^+}, \mathcal{O}(-1+n)) \quad \circledast$$

(a slight variation on the twistor case). Conjugation sends $\mathbb{CP}^{3+}$ to $\mathbb{CP}^{3-}$ (since ω is skew!) $\mathbb{CP}^3 = \times\!\Leftarrow\!\bullet$ (with a contact structure induced by ω); the role of minkowski space is played by $\bullet\!\Leftarrow\!\times = \mathbb{CS}^3$ (= 3 dim $\mathbb{C}$-minkowski space), regarded as a symplectic Grassmanian. [<u>Note</u>: $n = -1$: $\circledast$ is obviously an isomorphism, but the cohomology is not irreducible].

<u>The general Hermitean symmetric case</u> Both standard twistor theory & the example are instances of a general construction.

To see this in terms of ordinary twistor theory recall that we can think of $\mathbb{M}^+$ in two ways: (i) as a complex manifold = open $SU(2,2)$ orbit in $\mathbb{M}$, (ii) as the real homogeneous space $SU(2,2)/S(U(2)\times U(2))$. In the latter guise it is not immediately clear (i) is true. We can ask, for general real semi-simple Lie groups G_0 with maximal compact subgroups K_0 when G_0/K_0 is a complex manifold, i.e. an open G_0 orbit in a complex homogeneous space G/P. ANSWER: see table below (for G_0 classical). The example just given is of this form. QUESTION: Is there a Twistor transform for such G_0/K_0? I.e. can we find $Q, Q' \subset G$ st. G/Q & G/Q' are complex manifolds with a twistor transform between them? ANSWER: Yes (see table) & what is more in either there is always a natural $\mathbb{Z}$'s worth of line bundles $\mathcal{O}(k)$, as on $\mathbb{CP}^3$ etc; as in example, it often happens that $Q = Q'$.

TABLE:

G_0	K_0	G/P	G/Q, G/Q'	Twistor transform: $n>0$
$SU(p,q)$	$S(U(p)\times U(q))$	$p-1$, $q-1$	and	$H^{p-1}(\mathcal{O}(-p-n)) \cong H^{q-1}(\mathcal{O}(-q+n))$
$SO(2n+1,2)$	$U(1)\times SO(2n+1)$		(self dual)	$H^{p(p-1)/2}(\mathcal{O}(-2p+1-n)) \cong H^{p}(-1+n)$
$Sp(p,\mathbb{R})$	$U(p)$		(" ")	$H^{p-1}(\mathcal{O}(-p-1-n)) \cong H^{p-1}(-p+1+n)$
$SO(4p,2)$	$U(1)\times SO(4p)$		&	$H^{2p(2p-1)/2}(\mathcal{O}(-4p+2-n)) \cong H^{2p-1}(-2+n)$
$SO(4p+2,2)$	$U(1)\times SO(4p+2)$		(self dual)	$H^{(2p+1)p}(\mathcal{O}(-4p-n)) \cong H^{2p}(-2+n)$
$SO^*(2p)$	$U(n)$		(" ")	$H^{p-1}(-p-n) \cong H^{p-2}(-p+2+n)$

In all of these, there are two open G_0-orbits on G/Q (& Q') & the transform is between corresponding orbits. A relative cohomology form of elementary state exists & $G/Q \longrightarrow G/Q'$ under complex conjugation; so all the twistor arguments should go through to produce unitary representations of each G_0.

Thanks to MGE & ED.

Rob Baston

[1] MGE/RJB TN 20.
[2] " "The Penrose Transform — its interaction with representation theory" OUP. (1989).