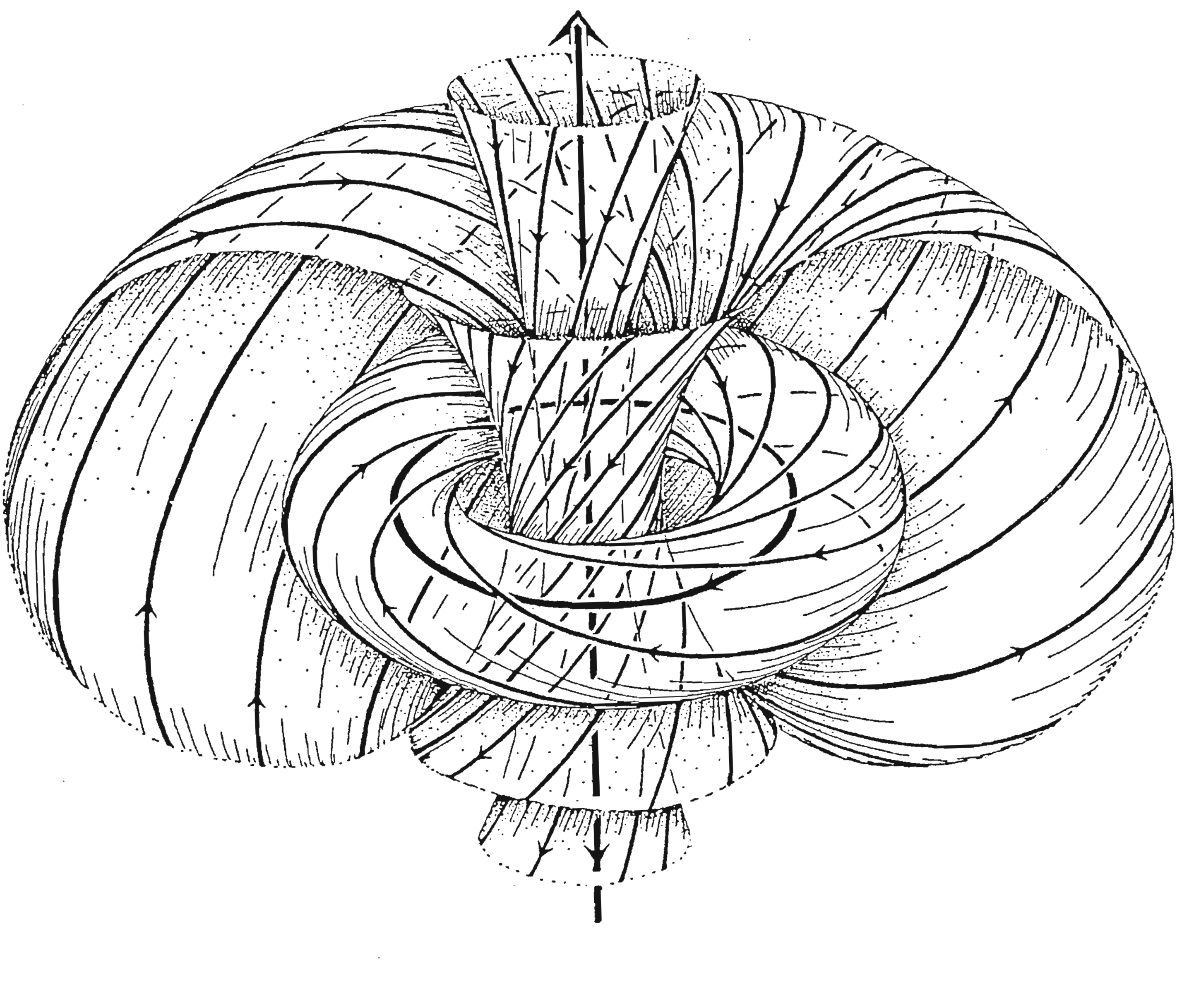
Twistor Newsletter (nº 27 : 7, Dec. 1988)



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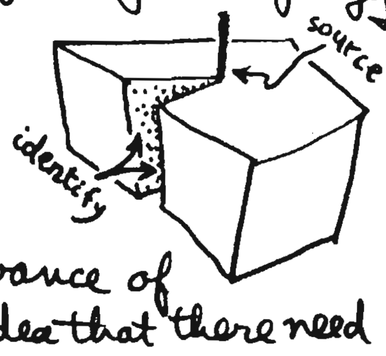
Topological QFT and Twistors: Holomorphic Linking

There is much current interest in the idea of "topological quantum field theories" (TQFT) particularly in relation to pure mathematical applications: Fleur cohomology, Donaldson invariants, Jones polynomials, etc. However the physical relevance of such "field theories" is more problematical. The salient feature of a TQFT is that there are no field equations — the dynamics vanishes altogether locally, and the theory can be formulated to be independent of the choice of metric. This is very attractive from the mathematical point of view – where one may be interested in global (differential) topological properties rather than properties depending upon a chosen metric or (physical) field. But physical relevance remains unclear. A noteworthy example, where field equations vanish locally, is provided by (2+1)-dimensional "gravity": in "vacuum" we have $R_{ab}=0$ whereas the Weyl curvature necessarily vanishes ($C_{abcd}=0$) in 3-dimensions [cf. E. Witten preprint "2+1 Dimensional Gravity as an Exactly Soluble System", Sept. 1988, IASSNS-HEP-88/32]. Thus the "vacuum space-time" is always flat ($R_{abcd}=0$), the (classical) interest coming from global topology and "deficit angles" as regions of "non-vacuum" (singular world-lines?) are circumnavigated.



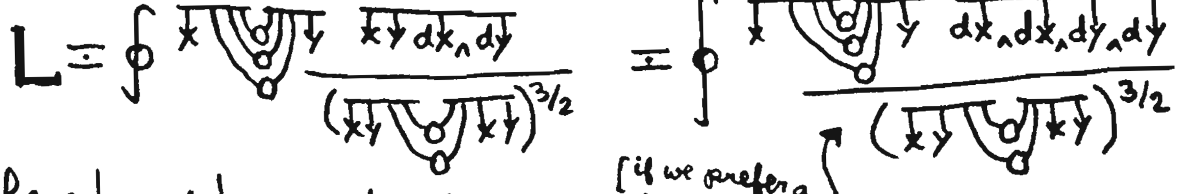
Twistor theory offers a very different viewpoint as to the possible physical relevance of a TQFT. We are familiar with the idea that there need be no local information stored in a deformed twistor space, or in a twistor space with a holomorphic bundle over it, or a 1-function (element of an H^1) defined on it. Thus, the physically appropriate home for TQFT might well be twistor space rather than space-time.

There is, of course, one fundamental difference between the envisaged setting for TQFT and twistor space: TQFTs refer to <u>real</u> manifolds whereas twistor spaces are <u>complex</u>. However, this need not be a drawback. Ideas can be carried over from the real to the (holomorphic-) complex whenever there is an <u>analytic</u> <u>integral formula</u> <u>expressing</u> what is needed. This can lead to some surprising applications.

Consider the ordinary <u>linking number</u> **L** for two closed curves in Euclidean 3-space. There is a formula (cf. Flanders, "Differential Forms", p. 80)

$$\mathbf{L} = \frac{1}{4\pi}\oint\!\!\oint \frac{(\vec{x}-\vec{y})\cdot d\vec{x}\wedge d\vec{y}}{|\vec{x}-\vec{y}|^3}$$

for this (expressing the work done on a magnetic monopole carried around one curve, where there is a fixed current in the other — or, no more physically, the work done on an electric charge carried around a "magnetic current"!). We want to interpret this formula as a complex contour integral over (contours in) two holomorphic curves in $\mathbb{CP}^3$, i.e. in $\mathbb{PT}$. (N.B. curves in $\mathbb{RP}^3$ can have quarter-integral linking numbers; but the following <u>might</u> encounter unwanted branching in the $(\ldots)^{3/2}$ in those cases.) We can rewrite **L** <u>projectively</u> (using the sign $\doteq$ to denote equality up to a multiplicative factor that I haven't yet bothered to work out) as



where x and y are twistors specifying points X and Y (not necessarily compact), respectively, each on the two curves varying along its appropriate (1-real-dim. closed) contour, and where γ specifies an arbitrarily chosen "metric" on $\mathbb{PT}$, either complex Euclidean, in which case (= 24 det(γ)) vanishes, or complexified sphere metric (det(γ) ≠ 0). This expression for **L** is independent of small finite deformations of

(a) the contours (obviously)
(b) the curves X and Y (not so ")
(c) the "metric" Ω (not so ").

These variations can take place within some neighbourhood $\mathcal{N}$ (a complex manifold) of the contours in $\mathbb{PT}$. We must make sure that the "distance" between x and y never becomes zero (i.e. $\overline{xy}\,\Omega\,\overline{xy} \neq 0$) within $\mathcal{N}$ (or, rather, on the contours), in our variation.

Note the curious fact that this "linking" between X and Y is <u>not</u> simply a <u>topological</u> property since real 2-surfaces in real 6-space don't link. It only works because we are allowing only <u>holomorphic</u> variations of X, Y, Ω.

The following expression, for which I am grateful for important assistance from A.P.H. $\rightsquigarrow$ $\displaystyle \frac{x\,\Omega\,y}{(\overline{xy}\,\Omega\,\overline{xy})^{3/2}} = \oint \frac{\mathcal{D}W \log[\,\Omega_{\psi\psi}/\Omega'_{\psi\psi}\,]}{(\psi_x\,\psi_y)^2}$

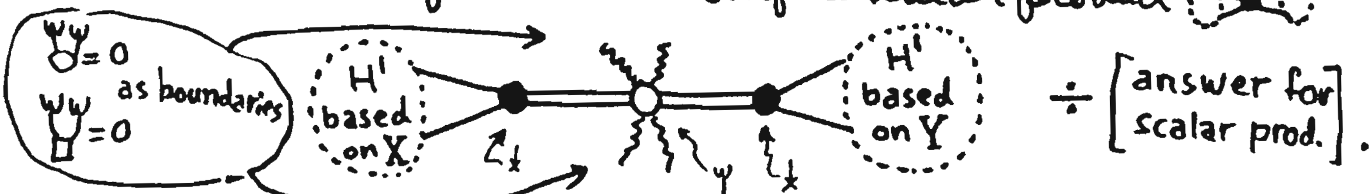
(Ω' "arbitrary") can be used to express $\mathbf{L}$ as:
(introduce new ψ)

$$\mathbf{L} = \oint_{\text{(proj.)}} \frac{\log[\,\Omega_{\psi\psi}/\Omega'_{\psi\psi}\,]\;\underline{\psi\, d\psi \wedge d\psi \wedge d\psi}\;\underline{xy \wedge dx \wedge dy}}{(\psi_x\,\psi_y)^2} = \oint_{\text{(non-proj.)}} \frac{\log[\,\Omega_{\psi\psi}/\Omega'_{\psi\psi}\,]\,d\psi \wedge d\psi \wedge d\psi \wedge d\psi \wedge dx \wedge dx \wedge dy \wedge dy}{(\psi_x\,\psi_y)^2}$$

(d^4W)

which is handy for examining its invariance properties.

A.P.H. points out that this expression is essentially a "double twistor transformed" version of the scalar product



Here the 1-functions are standard cohomology elements specifying the two curves X, Y (cf. R.P. in TN 4 ; M.G.E. & L.P.H. in TN 8 ; both reprinted in Adv. in Tw. Th., for the case of a twisted cubic). M.F.A. recently suggested to me a possible role, in this context, for such scalar product expressions (related to Green's functions), and I am grateful to him for valuable conversations.

Thanks are due also to A.P.H. and other members of the twistor group.

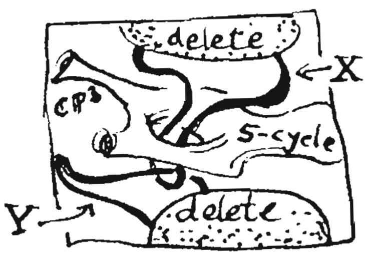
<u>QUESTION</u>: can <u>causality</u> <u>relations</u> between points in $\mathbb{M}$ be understood in terms of holomorphic linking of the corresponding $\mathbb{CP}$'s in $\mathbb{PT}$? How does this relate to Rob Low's thesis work?

~ Roger Penrose

Holomorphic Linking: Postscript

If we formally integrate out the $=\!\!\ast\!\!=$ in the double twistor transform $(X)\!\!\succ\!\bullet\!=\!\ast\!=\!\bullet\!\!\prec\!(Y)$ to get $(X)\!\!\times\!\!(Y)$, and represent each 1-function $(X)\!\!\succ$, $\prec\!(Y)$ locally by a pair of equations $f=0=g$, $h=0=j$, respectively, then we get $\mathbf{L} \mp \sum \oint \frac{f'(z)\, g'(z)\, h'(z)\, j'(z)\, d^4 z}{f(z)\, g(z)\, h(z)\, j(z)}$ ($f' = \frac{\partial f}{\partial z}$ etc.)

i.e. $\mathbf{L} \mp \sum \oint \frac{df \wedge dg \wedge dh \wedge dj}{f g h j} = \sum\left(\pm (2\pi i)^4\right)$, where the "$\Sigma$" tots up all the needed pieces for the two 1-functions. In the case where X and Y are algebraic curves, we can use a dot product of the two H^1's to get an H^3 elem. & evaluate this 3-function over $\mathbb{PT}$ to get a canonical value for $\mathbf{L}$ (i.e. indep. of a selection of contours). If we normalize $\mathbf{L}$ so that $\mathbf{L}=1$ for two lines in $\mathbb{CP}^3$, then we get $\boxed{\mathbf{L} = xy}$ where x and y are the orders of $\mathbf{X}$ and $\mathbf{Y}$, respectively (number of points they meet a generic plane). This is one quarter of the ("standard") linking number which would be **1** for two linking circles. (This result can be obtained by specializing $\mathbf{X}$ to x lines and $\mathbf{Y}$ to y lines.) A possible suggestion for the more general holomorphic linkings (e.g. distinguishing $\boxed{\text{OO}}$ from $\boxed{\text{linked circles}}$!) might be to find a real 5-cycle (in $\mathbb{CP}^3$) separating



$\mathbf{X}$ from $\mathbf{Y}$ and such that $\exists$ 1-functions defining $\mathbf{X}$ and $\mathbf{Y}$ whose domains intersect in a region containing the 5-cycle. Each 1-function is a 1-form projectively and a 2-form non-projectively. Use the non-projective form, and cup product to obtain a (2+2)-form (1+1)-function, i.e. element of $H^2(d\Omega^3)$ (since closed). We can evaluate this on the non-projective 6-cycle which is the circle bundle over the 5-cycle in the S^7 of "unit circles" over $\mathbb{CP}^3$ (in $\mathbb{C}^4$). I don't know if this gives significantly more than we had before.

REMARK I find the relation between holomorphic linking and twistor diagrams intriguing. Linking (at least, real linking) is a combinatorial thing. Could there be a deep combinatorial aspect to twistor diagram theory? Could this relate to the original motives lying behind spin-network theory?

~ Roger Penrose

CAUSAL RELATIONS AND LINKING IN TWISTOR SPACE

This is a brief review of some work on the relationship between the causal separation of points (mainly in Minkowski space) and the way in which their corresponding skies in (projective null) twistor space link. To fix notation, let M be Minkowski space, and let PN^I be projective null twistor space without I, so that PN^I is the space of null geodesics of M, with topology $\mathbb{R}^3 \times S^2$. If x is a point in M, then $X \cong S^2$ is its sky in PN^I.

Then let $x,y \in M$, and let S be a Cauchy surface containing y (for example, the surface of constant t). Given this, $PN^I \cong S \times S^2$, and so $PN^I \backslash I$ is diffeomorphic with $(S \backslash \{y\}) \times S^2$, which has the topology $(\mathbb{R} \times S^2) \times S^2$. A simple calculation using the Künneth sequence[1] shows that

$$\begin{aligned} H_2(PN^I \backslash Y) &\cong H_2(S \backslash \{y\}) \oplus H_2(PN^I) \\ &\cong \quad \mathbb{Z} \quad \oplus \quad \mathbb{Z} \end{aligned}$$

(where all homology is taken to be with coefficients in $\mathbb{Z}$). Thinking geometrically, then, if we consider the image of the fundamental class of X in $H_2(PN^I \backslash Y)$, we get a pair of integers, the first of which tells us how often X wraps round Y; this is used to motivate a definition of linking number for skies. First, however, the question of the orientation on a sky must be considered. To any sky in PN^I, associate the orientation which induces the inward pointing normal to $I^-(x) \cap S$ when S is a Cauchy surface just to the past of x.

Definition. Let X and Y be the skies of points of M such that $X \cap Y = \emptyset$. Then Link(X,Y) is defined to be the image of the fundamental class of X in $H_2(PN^I \backslash Y)/H_2(PN^I)$.

It follows that x and y are timelike separated if and only if $\mathrm{Link}(X,Y) \neq 0$. In fact, we have

Theorem. Let $x,y \in M$ with skies $X,Y \subset PN^I$ respectively, such that $X \cap Y = \emptyset$. Then

$$x \in I^{\pm}(y) \text{ if } \mathrm{Link}(X,Y) = \pm 1$$

$$x \in M \backslash J(y) \text{ if } \mathrm{Link}(X,Y) = 0. \qquad \square$$

Given that this version of linking seems to work so well, one is tempted to try to get the intersection theoretic version[2] of linking to work; in this case, one is immediately faced with the problem that the sky of a point is not the boundary of any surface in PN^I, so there is a problem caused by the topological non-triviality of PN^I. One attempt at circumventing this problem might be to consider surfaces in $PN\setminus Y$ with boundary $I\cup X$, corresponding to considering the homology of X in $PN\setminus Y$ relative to I. This cannot work, though, because these relative homology groups all vanish.

A slight subtlety involving the way in which the surface approaches X and I will recover the situation.

Theorem. Let $x,y\in M$ such that $X\cap Y=\emptyset$. Then if $\Sigma\subset PN$ is a surface with boundary $X\cup I$ such that

1) $\Sigma \cong S^2\times[0,1]$

2) $\Sigma\setminus I \cong S^2\times[0,1)$

and 3) Carrying the orientation from X to I along Σ gives the same orientation on I as that induced by completing a Cauchy surface of M and regarding PN as the tangent sphere bundle of the resulting S^3

then

$x\in I^{\pm}(y)$ if the intersection number of Σ with Y is ± 1

$x\in M\setminus J(y)$ if the intersection number is 0. □

Notes.

1) In dimensions other than 4, most of this carries straight across, except that for odd dimensions the nice relationship between the sign of the linking number and the information of which point is to the future of the other is lost.

2) The skies of M form a maximal family of S^2s in PN^I with the transitivity property that $\mathrm{Link}(X,Y)=\mathrm{Link}(Y,Z)=1 \Rightarrow \mathrm{Link}(X,Z)=1$.

3) If M is a conformally flat globally hyperbolic space-time with a

non-compact Cauchy surface, then $x \in I^+(y)$ if Link(X,Y)>0, $x \in I^-(y)$ if Link(X,T)<0, and $x \in M \setminus J(y)$ if Link(X,Y)=0.

4) In curved space-times, the above results still hold locally in the sense that given any point $x \in M$ x has a causally convex globally hyperbolic neighbourhood with Cauchy slice $\mathbb{R}^3$, and for y lying in this set we obtain exactly the same relationship between linking of skies in the corresponding space of null geodesics as for Minkowski space.

Thanks to R. Baston and R. Penrose.

Robert Low.

References.

[1] Greenberg and Harper, Algebraic topology: a first course.

[2] Rolfsen, Knots and Links.

Abstract

A twistor conformal field theory for four space-time dimensions

A. P. Hodges R. Penrose M. A. Singer

A definition is proposed for four-dimensional conformal field theory in which a class of complex 3-manifolds and holomorphic sheaf cohomology replace the Riemann surfaces and holomorphic functions of two-dimensional theory. Suggestions are made about the interpretation of such a theory, in terms of the interaction of fundamental particles, in relation to the theory of twistor diagrams, and the possibility of extending it to incorporate gravity.

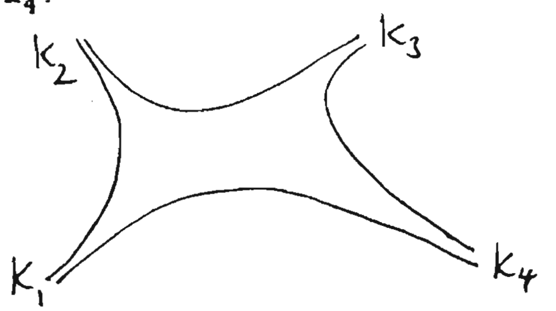
to appear in Physics Letters

Connections between amplitudes in string theory and twistor diagrams

In the proposal [1] for twistor conformal field theory in four dimensions (CFT4), an amplitude arises in the first instance as associated with a specific complex manifold $\mathbf{X}$. We then suggest that to obtain a conformally invariant amplitude there should be a summation of such amplitudes over (a class of) such complex manifolds. We hope to identify such conformally invariant amplitudes with the evaluations of twistor diagrams. Thus we are led to consider the possibility that at least some of the multiple integrations appearing in twistor diagrams are interpretable as integrations over manifold-defining parameters.

It is encouraging that the amplitudes calculated for the tree-diagrams of (bosonic) string theory can be put in the form of projective *spinor* integrals, interpretable in just this sense. This is only a matter of rewriting the calculations as given in Green, Schwarz and Witten, vol. 1, pp. 38-50, 355-390. As we shall see, a tantalising hint of an analogy with twistor diagrams then emerges.

The simplest example is that of integrating over the sheets formed by four open strings with spin 0, labelled by [26-dimensionall] momenta k_1, k_2, k_3, k_4.



$(k_1 + k_2 + k_3 + k_4 = 0)$

The argument is that first, such a sheet may be mapped conformally to a disc with four points on the boundary removed (equivalently, the upper-half-plane with four real points removed). Thus summation over all strings reduces to summing over conformally inequivalent discs (half-planes). This in turn reduces to a single integral over the cross-ratio of the four removed points, the range of integration being determined by the ordering (1234), well-defined up to cyclic permutation. This integral then produces the Veneziano beta-function appropriate to that ordering. Explicitly, using the upper-half-plane formulation, G. S. & W. write down

$$\int dz_1 dz_2 dz_3 dz_4 \; \exp(-k_1.k_2 \log|z_1-z_2|) \exp(-k_1.k_3 \log|z_1-z_3|)$$
$$\exp(-k_1.k_4 \log|z_1-z_4|) \exp(-k_2.k_3 \log|z_2-z_3|)$$
$$\exp(-k_2.k_4 \log|z_2-z_4|) \exp(-k_3.k_4 \log|z_3-z_4|)$$

where the integrand (i.e. the amplitude associated with a specific choice of z_1 , z_2 , z_3 , z_4) is deduced from an action principle. They arrive at an integral over the cross-ratio by imposing $z_1 = 0, z_3 = 1, z_4 = \infty$, (thought of as "gauge-fixing") so that the integral reduces to

$$\int_0^1 dz_2 \, |z_2|^{-k_1 . k_2} \, |1 - z_2|^{-k_2 . k_3}$$

k

They have a argument to show that this is SL2C invariant iff the external momenta meet the "tachyon" conditions

$$-k_1^2 = k_1 . k_2 + k_1 . k_3 + k_1 . k_4 = 2 \quad \text{(in units of the Planck mass)}$$

and similarly for k_2, k_3, k_4.

Writing $s = (k_1 + k_2)^2 \qquad t = (k_1 + k_4)^2 \qquad u = (k_1 + k_3)^2$
these imply $s + t + u = -8, \quad k_1 . k_2 = (s + 4)/2$ etc.

However this invariance is (to me) more transparent and symmetrically expressed when the amplitude is written in terms of projective spinors The Green function $\log|z_1 - z_2|$ then appears as

$$\log \left| \frac{z_1 . z_2}{z_1 . \alpha \;\; z_2 . \alpha} \right|$$

where the spinor α corresponds to the point at infinity. SL2C invariance, i.e. independence of α , is then obviously equivalent to the "tachyon" mass condition. If this is met, the integral then becomes

$$\int Dz_1 \ldots Dz_4 \left[(z_1 . z_2)(z_3 . z_4)\right]^{-a} \left[(z_1 . z_4)(z_2 . z_3)\right]^{-b} \left[(z_1 . z_3)(z_2 . z_4)\right]^{-c}$$

where $a = k_1 . k_2 = 2 + s/2; \quad b = k_1 . k_4 = 2 + t/2; \quad c = k_1 . k_3 = 2 + u/2,$

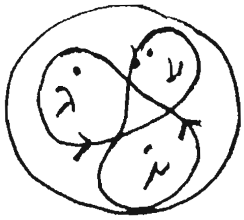
so $\qquad a + b + c = 2.$

and where the integration can be "freed" from the real line.

Such an integrand is familiar in spinor integral calculus. Note the symmetric contour integral formula

$$\frac{1}{(2\pi i)^2}\oint_C D\pi\,(\pi.\lambda)^{-a}(\pi.\mu)^{-b}(\pi.\nu)^{-c} = \frac{(\lambda.\mu)^{c-1}(\lambda.\nu)^{b-1}(\mu.\nu)^{a-1}}{\Gamma(a)\Gamma(b)\Gamma(c)}$$

(a + b + c = 2)

where C is the compact contour  in **CP**[1]

C is equivalent to a Pochhammer contour winding round a cut between two branch points, and so this compact contour integral formula implies the non-compact "beta-function" integral formula

$$\int_\lambda^\mu D\pi\,(\pi.\lambda)^{-a}(\pi.\mu)^{-b}(\pi.\nu)^{-c} = \frac{\pi}{\sin\pi a}\,\frac{\pi}{\sin\pi b}\oint_C (\pi.\lambda)^{-a}(\pi.\mu)^{-b}(\pi.\nu)^{-c}\,D\pi$$

$$= \frac{\Gamma(1-a)\,\Gamma(1-b)}{\Gamma(2-a-b)}\,(\lambda\mu)^{c-1}(\lambda.\mu)^{b-1}(\mu.\nu)^{a-1}$$

If we integrate out z_2 in this way (following G. S. and W.) we are left with

$$\frac{\Gamma(1-a)\Gamma(1-b)}{\Gamma(2-a-b)}\int Dz_1\,Dz_2\,Dz_3\;(z_1.z_3)^{-1}(z_3.z_4)^{-1}(z_4.z_1)^{-1}$$

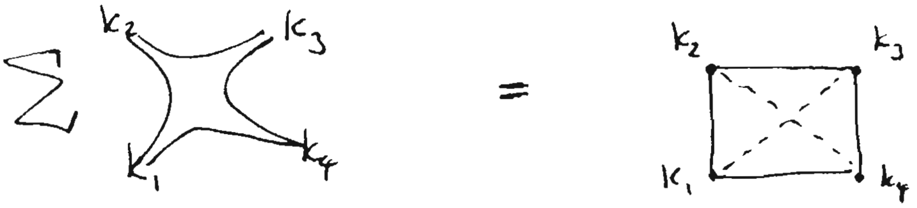
To G. S. and W. this remaining integral is a divergent integral which represents the volume of SL2R, and is to be divided out. As an alternative way of expressing this idea, we may change the contour to a compact $S^2 \times S^1$ and obtain the required finite result. Putting these ideas together, we may claim that the ***compact spinor contour integral***

$$\oint Dz_1\ldots Dz_4\;\frac{[(z_1.z_2)(z_3.z_4)]^{-k_1.k_2}}{\sin\pi(k_1.k_2)}\;\frac{[(z_1.z_4)(z_2.z_3)]^{-k_1.k_4}}{\sin\pi(k_1.k_4)}\;[(z_1.z_3)(z_2.z_4)]^{-k_1.k_3}$$

corresponds to an invariantly defined summation over strings, yielding the Veneziano amplitude

$$A(s,t) = \frac{\Gamma(-1-\frac{1}{2}s)\,\Gamma(-1-\frac{1}{2}t)}{\Gamma(-2-\frac{1}{2}s-\frac{1}{2}t)}$$

We might express this spinor integral by a "spinor diagram", so that

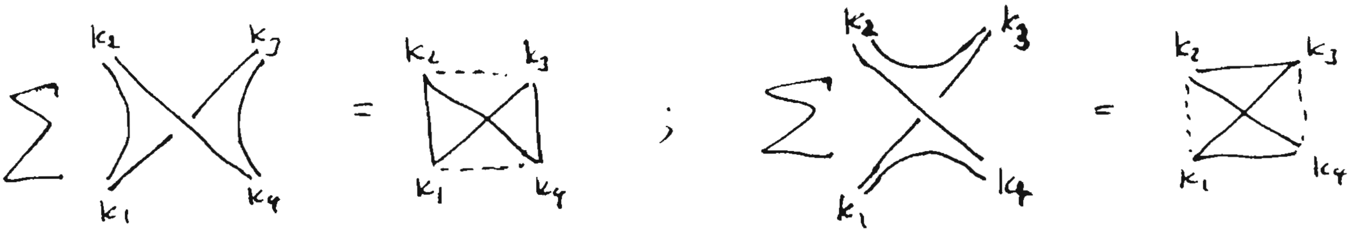


Here the solid lines represent factors $\dfrac{[(z_1.z_2)(z_3.z_4)]^{-k_1.k_2}}{\sin\pi(k_1.k_2)}$, $\dfrac{[(z_1.z_4)(z_2.z_3)]^{-k_1.k_4}}{\sin\pi(k_1.k_4)}$

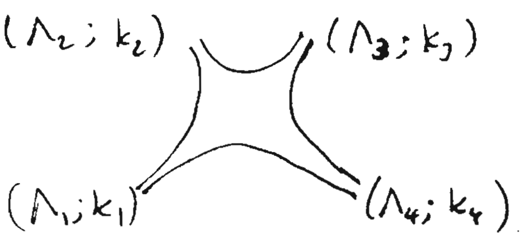
the dashed line a factor $[(z_1.z_3)(z_2.z_4)]^{-k_1.k_3}$ (a *period* of the solid line)

and compact contour integration over the z_i is implied.

By permuting the external states we have likewise

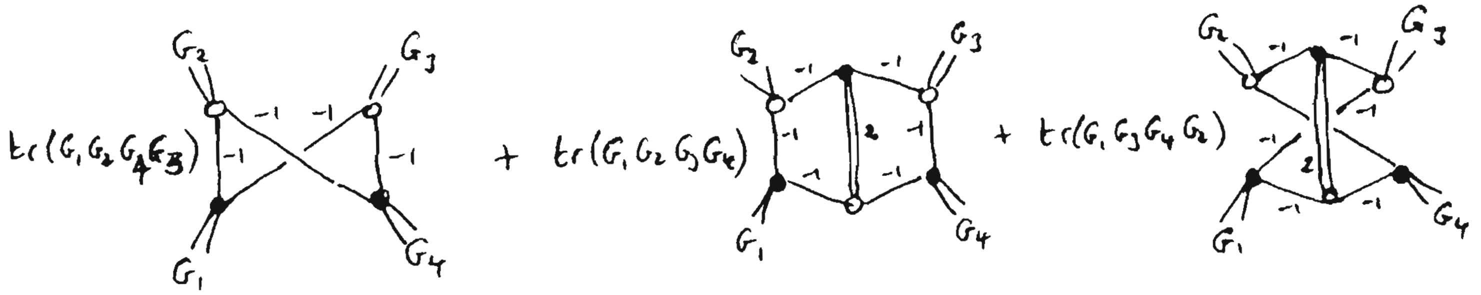


The significance of the ordering of the external states is seen in its connection with internal symmetry. Suppose the external states now also carry elements Λ_i of U(n) as internal symmetry indices (such an element can be thought of as describing a quark-antiquark pair in the hadron model for which the original bosonic string theory was developed). Then according to string theory the amplitude for

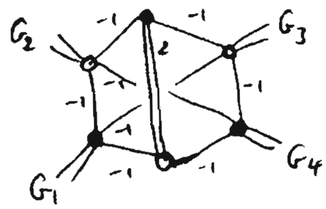


is assigned a coefficient of: $\mathrm{tr}(\Lambda_1\Lambda_2\Lambda_3\Lambda_4)$

Now it is very striking that such a trace structure, respecting order up to cyclic permutation, has already been observed in twistor diagram theory. As described in TN 23, the amplitude for pure SU(2) gauge field scattering takes the form



Note that these twistor integrals fall into a form analogous to the spinor integrals above, since the period of the logarithmic (-1)-line is just unity, i.e. each is a "period" of



Of course these integrals are entirely different from the spinor integrals above in that the external state parameters appearing in the exponents now correspond to ***helicities*** and not to ***momenta*** Furthermore we have no action principle or Green function in the twistor picture to lend substance to these similarities.

But the analogy is close enough to give some more support to our conjecture that there exists an interpretation of these integrals as suitable invariant summations over twistor manifolds within a CFT4.

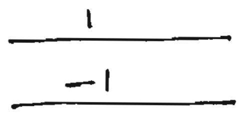
Andrew Hodges

[1] A.P.H., R.P. and M.A.S., to appear in *Physics Letters*, see also TN 26.

Notes on the correspondence between Feynman diagrams and twistor diagrams

In **TN25** we established a remarkable twistor-diagrammatic formula for second-order massless ϕ^4 scattering and deduced (a) some extensions to higher-order ϕ^4 amplitudes and (b) a new viewpoint on the crossing symmetry problem for the first-order ϕ^4 amplitude. Here are some notes on the features that arise on attempting to develop these ideas more generally and systematically as a "translation" procedure.

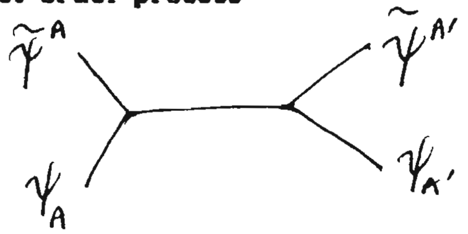
§1. One clear feature emerging from these ϕ^4 diagrams is the pair of lines



representing the "off-shell" scalar propagator $\Delta_F(x - y)$

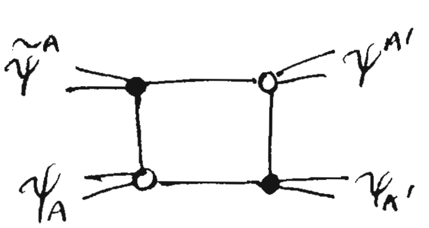
with the "on-shell" propagator  as its "period".

Note that this scalar propagator can be exhibited in a simpler context, namely the first-order process



where the vertex is given by the so-called "Yukawa interaction" of massless fields expressed by the Lagrangian $\tilde{\psi}^A \psi_A \phi$

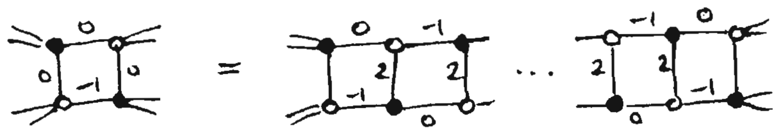
Standard arguments translate this first-order amplitude into

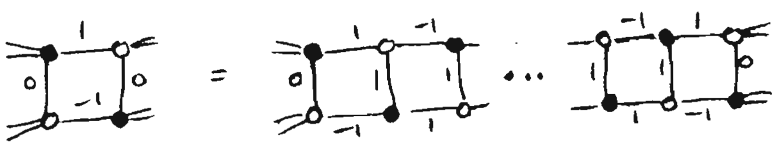


One can think of the massless spin-1/2 fields as *test-functions* for the twistor representation of $\Delta_F(x - y)$. It is generally useful and important to include the conformally invariant "Yukawa" interaction along with the ϕ^4 interaction in studying the structure of higher order diagrams.

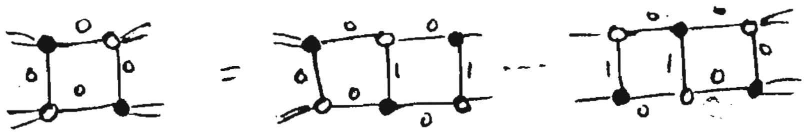
§2. The scalar propagator then fits into a scheme together with

for massless spin-1/2, spin-1 propagators respectively. In each case the propagator can be regarded as an operator which projects out an eigenstate of spin, in such a way that the operator is idempotent. Thus:





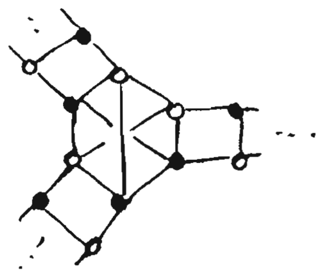
The contours here are consistent with



If Feynman propagators Δ_F are equivalent to these "chains" then how could Feynman *vertices*, of essential form

$$\int \Delta_F(x-x_1)\ \Delta_F(x-x_2)\ \Delta_F(x-x_3)\ d^4x\,,$$

be translated? One might expect twistor-diagram vertices of form:



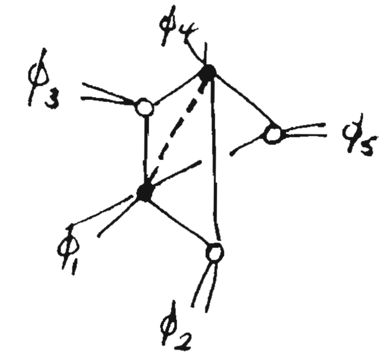
For this to be true, a necessary condition is that we can differentiate w.r.t. x_1, x_2 and x_3, thus reducing each Δ_F to the δ-function, and get the right answer - as evaluated by using six massless fields [two in x_1, two in x_2, two in x_3] as "test-functions". The form of the derivatives will depend upon which particular helicities we are looking at, but essentially the problem boils down to establishing a twistor-diagrammatic formula, taking this form, for the scalar ϕ^6 integral

$$\int \phi_1(x)\,\phi_2(x)\,\phi_3(x)\,\phi_4(x)\,\phi_5(x)\,\phi_6(x)\ d^4x$$

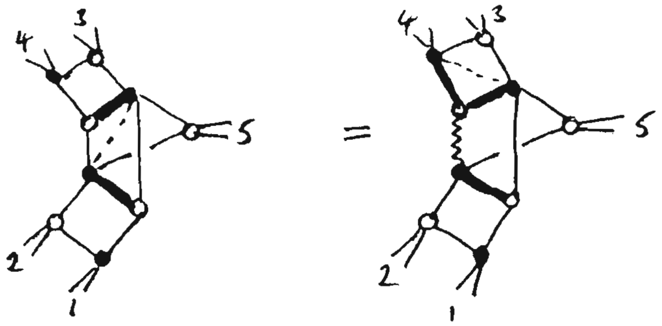
§3. There *is* a naturally suggested structure of this form for the ϕ^6 integral. It's useful here to use an abbreviated twistor diagram notation:

write  for 

Now let us begin with ϕ^5. Certainly the diagram

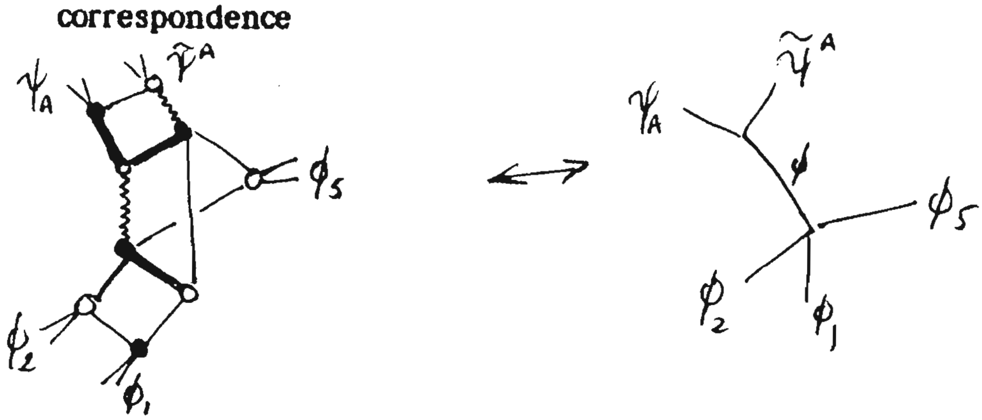


represents the ϕ^5 integral in one channel (235-14). The guess is that

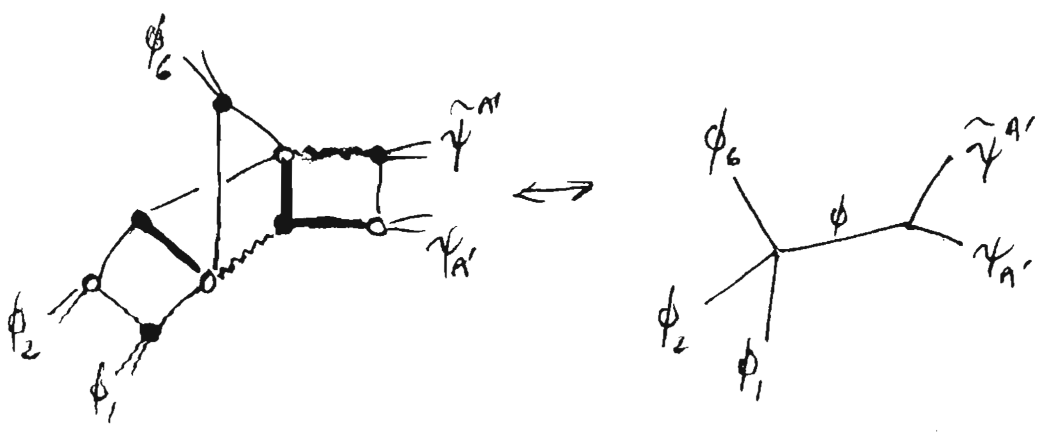


(using integration by parts)

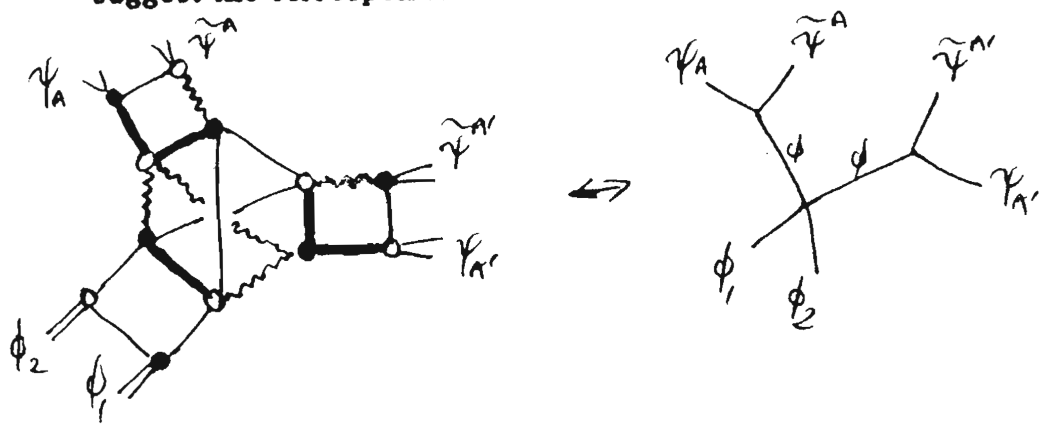
is valid in all channels. This would imply the (conformally invariant) correspondence



(again using simple integration by parts). If so we have also

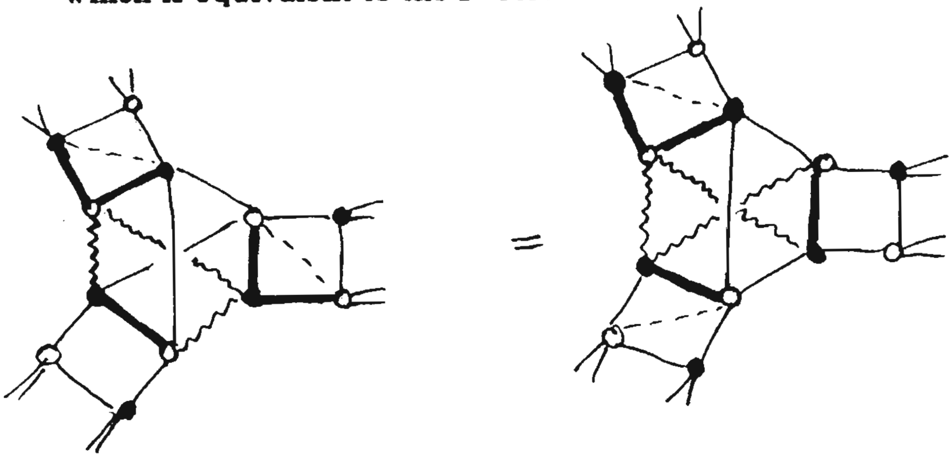


By "superimposing" these (i.e. looking at period structure) we are led to suggest the correspondence



(*)

which is equivalent to the ϕ^6 formulas



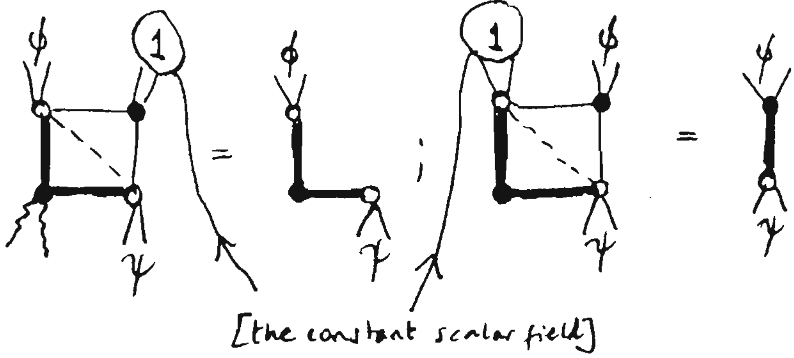
etc. etc.

(by permutation symmetry). This is in turn is the key to describing all Feynman vertices in twistorial terms, as indicated in §1.

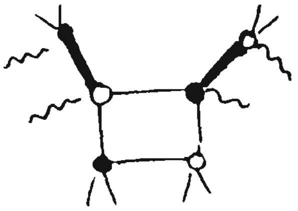
These are only guesses, but they embody important constraints on what formulas could possibly be correct. For any possible ϕ^6 formula implies a formula for the Feynman diagram (*), which one would expect (at least naively; see below) to be conformally invariant, and to have periods related to known integrals. What we have done above is to construct formulas which have a chance of satisfying these strong conditions. Unfortunately they have so far proved too difficult to check explicitly.

Another highly significant constraining feature of these integrals arises from the limiting cases in which one external massless field is allowed to tend towards the *constant* field. In the twistor picture this is a geometrical limit arising as the point defining an elementary state moves towards I. In the space-time integrals the effect is that of reducing ϕ^6 to ϕ^5 or ϕ^5 to ϕ^4. These limits must all make sense and agree - a strong condition.

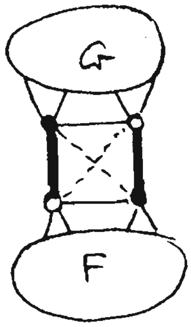
In particular we require contours which justify:



with the numerator factor involving $I^{\alpha\beta}$ "cancelling" the effect of the external parameters tending to $I^{\alpha\beta}$. At first sight, this is impossible: integration by parts seems to show that the limit has to be zero. But it *does* seem to be possible if the external fields are attached in the way suggested in **TN25**, viz. as



This problem is connected with the puzzle noticed long ago by RP, namely that one may formally deduce



for the inner product of two 2-twistor functions of mass zero, and yet simple integration by parts shows that such an integral must vanish.

A further question concerns the assumption made above that higher-order diagrams built out of conformally invariant interactions, should themselves be conformally invariant functionals. In standard QFT conformal symmetry breaking is necessary for renormalisation and so it is possible that in the twistor picture also conformal symmetry breaking has to play some essential role in building up diagrams.

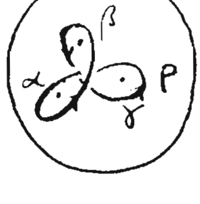
Putting these points together, it looks very likely that the formulae suggested above cannot actually make sense without some modification of the geometry of twistor space at I - something expected anyway for the description of mass (and gravity). My hope is that when correctly interpreted, the twistor diagrams written down above will form the basis of a systematic calculus.

A. P. Hodges

A note on Pochhammer contours

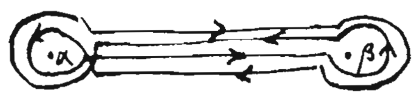
The formula

$$\oint_P \frac{D\pi}{(\pi.\alpha)^a(\pi.\beta)^b(\pi.\gamma)^c} = \frac{(2\pi i)^2(\alpha.\beta)^{c-1}(\beta.\gamma)^{a-1}(\gamma.\alpha)^{b-1}}{\Gamma(a)\,\Gamma(b)\,\Gamma(c)}$$



$(a + b + c = 2)$

can be demonstrated by deforming the symmetrical contour P to



Provided Re(b) and Re(c) are both > -1, this integral is:

$(1 - e^{2\pi ia})(1 - e^{2\pi ib}) \times$ (standard noncompact beta-function integral)

+ (contributions from small arcs vanishing as radii tend to 0)

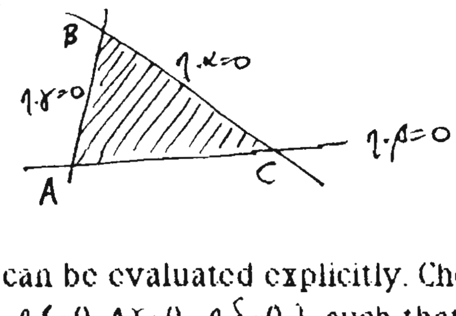
Extension to all non-integral a, b, c then follows by analytic continuation.

This formula and its justification generalises to $\mathbf{CP}^2$ (and presumably to $\mathbf{CP}^n$); namely we have for a 2-dimensional Pochhammer contour P,

$$\oint_P \frac{D^2\eta}{(\eta.\alpha)^a(\eta.\beta)^b(\eta.\gamma)^c(\eta.\delta)^d} = \frac{(2\pi i)^3(\overline{\alpha\beta\gamma})^{d-1}(\overline{\beta\gamma\delta})^{a-1}(\overline{\gamma\delta\alpha})^{b-1}(\overline{\delta\alpha\beta})^{c-1}}{\Gamma(a)\,\Gamma(b)\,\Gamma(c)\,\Gamma(d)}$$

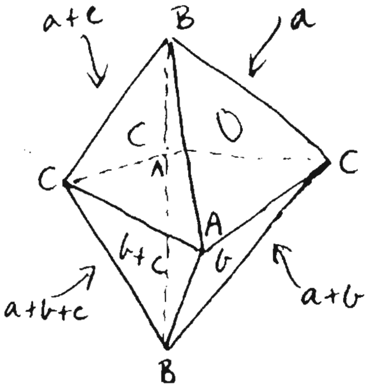
$(a + b + c + d = 3)$

In this case, to define the contour P, consider the noncompact integral of the given form over a triangular region:



The integral can be evaluated explicitly. Choose coordinates u, v for $\mathbf{CP}^2 - \{\eta\alpha=0, \eta\cdot\beta=0, \eta\cdot\gamma=0, \eta\cdot\delta=0\}$ such that $u = \eta\cdot\alpha/\eta\cdot\delta$, $v = \eta\cdot\beta/\eta\cdot\delta$
The resulting integral, a standard extension of the beta function integral, is finite provided Re(a), Re(b), Re(c) > -1.

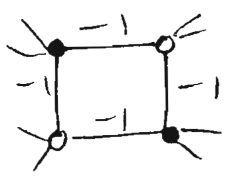
Now, we need to glue together eight copies of this triangle, together with sections with radius ε round the branch points.
One may check that the triangles fit togther like the faces of an octahedron, yielding a closed contour of topology S^2, and yield the integral as stated.



Triangular face labelled "b" indicates that the triangle is on the sheet characterised by factor $e^{2\pi ib}$ (etc.)

Special cases when one (or more) of the complex exponents is actually an integer. In the original one-dimensional case, the Pochhammer contour of topology S^1 becomes equivalent to the sum of two disjoint S^1 's (i.e. to $S^0 \times S^1$).
In the two dimensional integral what happens is that one "vertex" of the octahedron can be identified with the opposite vertex, so that the S^2 can be deformed into a torus $S^1 \times S^1$. Or this can be seen directly from the form of the integral: if d is an integer there is clearly a contour constructed as (small circle round $\eta \cdot \delta = 0$) x (one-dimensional Pochhammer contour inside the $\mathbf{CP}^1$ ($\eta \cdot \delta = 0$)).

Higher-dimensional Pochhammer contours of this kind play an essential role in evaluating such twistor diagrams as



where many logarithmic factors are to be convoluted.

Andrew Hodges

Cohomology Reduction through Qadir's Intertwining Operators

In his article on "Qadir's intertwining operators" (TN 25), R.P. described certain twistor contour integrals first suggested by A. Qadir in his Ph.D. thesis (London, 1970). The original use of these operators was to pass from representations of $SU(2,2)$ etc. to equivalent ones. Here, I wish to describe their cohomology reducing properties.

Qadir's integrals

Qadir's integrals (short QI) deal with holomorphic functions in four twistor variables W, X, Y, Z, homogeneous of degrees w, x, y, z respectively. Additionaly, these functions must satisfy

$$(1)\quad f(W+\lambda X+\mu Y+\gamma Z, X+\rho Y+\sigma Z, Y+\tau Z, Z) = f(W,X,Y,Z) \qquad (\forall\ \lambda,\mu,\gamma,\rho,\sigma,\tau).$$

In their original form, QIs are formal contour integrals

$$(2)\qquad \begin{aligned} a)\quad & g_1(W,X,Y,Z) = \oint f(W,X,Z,Y+\lambda Z)\,d\lambda \\ b)\quad & g_2(W,X,Y,Z) = \oint f(W,Y,X+\lambda Y,Z)\,d\lambda \\ c)\quad & g_3(W,X,Y,Z) = \oint f(X,W+\lambda X,Y,Z)\,d\lambda \end{aligned}$$

and some higher versions. Each expression satisfies (1) and is homogeneous of degrees

$$g_1 : (w,x,z+1,y-1) \qquad g_2 : (w,y+1,x-1,z) \qquad g_3 : (x+1,w-1,y,z).$$

We restrict ourselves to the case that w,x,y,z are integers. Condition (1) and homogeneity imply that f can be rewritten as a function of higher valence skew simple twistors:

$$F(\overline{WXYZ},\overline{XYZ},\overline{YZ},\overline{Z}) := f(\overline{W},\overline{X},\overline{Y},\overline{Z})$$

F is also homogeneous, of degrees $(w, x-w, y-x, z-y)$ in the respective variables. Hence, F actually represents a section of some line bundle over F_{123}, the flag manifold of projective twistor space (points $\subset$ lines $\subset$ planes $\subset$ $\mathbf{PT}$). The dependence of F on the first variable is in a sense trivial; indeed, the value of the first homogeneity degree does not alter the line bundle F represents. Therefore, the first variable is ignored in what follows.

We denote the sheaf of sections of this line bundle by $\mathcal{O}_{(x-w,z-y)[y-x]}$. Abstractly,

$$\mathcal{O}_{(p,r)[q]} := \eta_3^*\mathcal{O}(p)\otimes\eta_1^*\mathcal{O}(r)\otimes\eta_2^*\mathcal{O}[q]$$

where η_i^* are pullback maps for the fibrations $\eta_i : \mathsf{F}_{123} \longrightarrow \mathsf{F}_i$.
($\mathsf{F}_1 = \mathbf{PT}$, $\mathsf{F}_2 = \mathbf{M}$, $\mathsf{F}_3 = \mathbf{PT}^*$).

Cohomology reduction

The aim of this article is to show that there are isomorphisms

$$
(3)\qquad
\begin{array}{lll}
a) & H^n(\mathsf{F}_{123}, \mathcal{O}(p,-r-2)[q]) \xrightarrow{\simeq} H^{n-1}(\mathsf{F}_{123}, \mathcal{O}(p,r)[q-r-1]) & (r \geqslant 0) \\
b) & H^n(\mathsf{F}_{123}, \mathcal{O}(-p-2,r)[q]) \xrightarrow{\simeq} H^{n-1}(\mathsf{F}_{123}, \mathcal{O}(p,r)[q-p-1]) & (p \geqslant 0) \\
c) & H^n(\mathsf{F}_{123}, \mathcal{O}(p,r)[-q-2]) \xrightarrow{\simeq} H^{n-1}(\mathsf{F}_{123}, \mathcal{O}(p-q-1,r-q-1)[q]) & (q \geqslant 0)
\end{array}
$$

which are realised by the QIs a), b) and c) respectively.

The most interesting of these is case b); the others behave similarly. The proof of b) involves direct images along the fibration

$$\mu : \mathsf{F}_{123} \to \mathsf{F}_{13} = \mathsf{A}.$$

We require an interesting non-standard vector bundle over A, which can best be described by using a third spinor-type index.

DEFINITION . *The bundle $S_{\hat{A}}$ on A is given by its fibre at $[W_\alpha, Z^\beta] \in \mathsf{A}$*

$$S_{\hat{A}}\big|_{[W_\alpha, Z^\beta]} := \left\{\text{simple skew } R^{\alpha\beta} \text{ such that } W_\alpha R^{\alpha\beta} = 0 = R_{\alpha\beta} Z^\beta\right\}.$$

As for the usual spinors, there is a skew product on $S_{\hat{A}}$:

$$S_{\hat{A}} \wedge S_{\hat{B}} \cong S(-1,-1)$$

This is given explicitly by $(P^{\alpha\beta}) \wedge (R^{\gamma\delta}) \mapsto P^{\alpha\beta} R_{\beta\gamma}$ on the fibre over $[W_\alpha, Z^\beta]$. To see that this gives the desired result, write $P^{\alpha\beta} = Z^{[\alpha} X^{\beta]}$ and $R_{\beta\gamma} = Y_{[\beta} W_{\gamma]}$. X and Y must satisfy the incidence relations $Z^\alpha Y_\alpha = 0 = W_\alpha X^\alpha$, and therefore

$$P^{\alpha\beta} R_{\beta\gamma} = \tfrac{1}{4} Z^\alpha W_\gamma X^\beta Y_\beta \in \langle W_\gamma \rangle \otimes \langle Z^\alpha \rangle = S(-1,-1)\big|_{[W_\gamma, Z^\alpha]}$$

as required. As usual, the sheaf of sections of $S_{\hat{A}}$ is denoted by $\mathcal{O}_{\hat{A}}$.

We also need some vanishing statements (similar to the ones given in [EPW]).

VANISHING LEMMA. *For $\mu : \mathsf{F}_{123} \to \mathsf{F}_{13}$,*

$$\mu^i_* \mathcal{O}_{\hat{A}\ldots\hat{B}}(k,l)[m] = 0 \qquad \begin{cases} \forall\, i \neq 0 & \text{if } m \geqslant 0 \\ \forall\, i & \text{if } m = -1 \\ \forall\, i \neq 1 & \text{if } m \leqslant -2 \end{cases}$$

Fibres of μ are $\mathbf{CP}_1$, and direct images give the fibrewise cohomology, and so the lemma follows from the usual vanishing theorems for cohomology of homogeneous sheaves on $\mathbf{CP}_1$.

The proof of b) starts with the short exact sequence

$$\mathsf{F}_{123}: \quad 0 \to \mathcal{O}(p,r)[-q-2] \xrightarrow{\pi_{\hat{A}} \cdots \pi_{\hat{B}}} \mathcal{O}_{\underbrace{(\hat{A} \ldots \hat{B}\hat{C})}_{q+1}}(p,r)[-1] \xrightarrow{\pi^{\hat{C}}} \mathcal{O}_{\underbrace{(\hat{A} \ldots \hat{B})}_{q}}(p-1,r-1)[0] \longrightarrow 0$$

Multiplication by $\pi_{\hat{A}}$ is interpreted as the natural inclusion $\mathcal{O}[-1] \hookrightarrow \mathcal{O}_{\hat{A}}$ (on F_{123}), and $\pi^{\hat{C}} = \epsilon^{\hat{C}\hat{D}} \pi_{\hat{D}}$ requires the symplectic form $\epsilon^{\hat{C}\hat{D}} \in \mathcal{O}^{[\hat{C}\hat{D}]}(-1,-1)$. Taking cohomology along fibres of μ gives a long exact sequence of direct image sheaves. Because of the vanishing lemma, the only nonzero part of this long exact sequence is the coboundary map

$$\mu_* \mathcal{O}_{(\hat{A} \ldots \hat{B})}(p-1,r-1)[0] \xrightarrow{\simeq} \mu^1_* \mathcal{O}(p,r)[-q-2].$$

Now we can use

$$\begin{aligned} \mathsf{F}_{13}: \qquad & \mathcal{O}^{(\hat{A} \ldots \hat{B})} \cong \mu_* \mathcal{O}[q] \\ & \Longrightarrow \mathcal{O}_{(\hat{A} \ldots \hat{B})} \cong \mu_* \mathcal{O}(-q,-q)[q] \end{aligned}$$

to get

$$\mu^1_* \mathcal{O}(p,r)[-q-2] \cong \mu_* \mathcal{O}(p-1-q, r-1-q)[q]. \tag{4}$$

Finally, the move back to F_{123} is accomplished with the Leray spectral sequence:

$$E_2^{p,q} = H^q(\mathsf{F}_{13}, \mu^p_* \mathcal{F}) \Longrightarrow H^{p+q}(\mathsf{F}_{123}, \mathcal{F}) \qquad \mathcal{F} \text{ sheaf on } \mathsf{F}_{123}.$$

For both sheaves involved in (4), the corresponding spectral sequences are degenerate (again by the vanishing lemma). Hence

$$\begin{aligned} H^n(\mathsf{F}_{123}, \mathcal{O}(p,r)[-q-2]) &\cong H^{n-1}(\mathsf{F}_{13}, \mu^1_* \mathcal{O}(p,r)[-q-2]) \\ &\cong H^{n-1}(\mathsf{F}_{13}, \mu_* \mathcal{O}(p-q-1, r-q-1)[q]) \\ &\cong H^{n-1}(\mathsf{F}_{123}, \mathcal{O}(p-q-1, r-q-1)[q]). \end{aligned} \tag{5}$$

as desired. □

Starting with (3), one can go on to prove the Borel–Weil theorem on F_{123} by stringing isomorphisms together.

Now, where do QIs come in? Again, take case b) in (2) as an example:

$$G(\overset{++++}{XYZ}, \overset{+++}{YZ}, \overset{|}{Z}) = \oint F(\overset{++++}{XYZ}, \overset{+++}{XZ} + \lambda \overset{+++}{YZ}, \overset{|}{Z})\, d\lambda \tag{6}$$

in F_{123}-compatible notation. The integration takes place over branched contours on the fibres of $\mu : \mathsf{F}_{123} \to \mathsf{F}_{13}$ (the point $\lambda = \infty$ must be included), and the homogeneity changes from $(p, -q-2, r)$ in F to $(p-q-1, q, r-q-1)$ in G.

Given a Stein neighbourhood U in F_{13}. Then, (6) maps first Čech cohomology representatives on $\mu^{-1}(U)$ to global functions on $\mu^{-1}(U)$ and preserves cohomology classes. In other words, we have a map

$$\oint: \quad \mu^1_* \mathcal{O}(p,r)[-q-2](U) \xrightarrow{\simeq} \mu_* \mathcal{O}(p-1-q, r-1-q)[q](U). \tag{7}$$

This involves picking a (suitably nice) Stein cover of $\mu^{-1}(U)$, and choosing a (smooth) family of branched contours (one on each fibre over U) compatible with this cover. Roughly, this means that each double overlap of the cover (restricted to a fibre) contains exactly one branch. (Branched contours are explained in detail in "Spinors and Spacetime II"). Then, a splitting process due to Spaling and Ward (See **TN** 1) is performed to give the result.

All this is just a generalisation of the usual zero rest mass integral procedure.

Anyway, by choosing the U's to be the n-fold overlaps of a (nice) Stein cover of F_{13}, (7) induces the map

$$\oint: \quad H^{n-1}(\mathsf{F}_{13}, \mu^1_* \mathcal{O}(p,r)[-q-2]) \xrightarrow{\simeq} H^{n-1}(\mathsf{F}_{13}, \mu_* \mathcal{O}(p-q-1, r-q-1)[q])$$

which realizes the middle step of (5).

Many thanks to Roger Penrose and Rob Baston for lots of discussions and suggestions.

Klaus Pulverer

References:

R.P., "On Qadir's intertwining operators", **TN** 25.
R.P., "Relativistic symmetry groups"
in *Group theory in non-linear problems* (ed. A. O. Barut), Reidl, 1974.
[EPW] = M.G.E., R.P. and R.O.W., "Cohomology and massless fields",
Comm. Math. Phys. 78

Coming soon....

"The Penrose Transform: Its Interaction with Representation Theory"

by M. G. Eastwood and R J Baston.

Oxford University Press. 1989.

NON-HAUSDORFF TWISTOR SPACES FOR KERR AND SCHWARZSCHILD

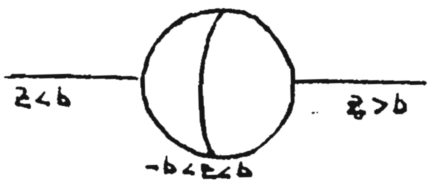
It is well-known that Einstein's equations for stationary, axi-symmetric vacuum space-times can be reduced to a form of the rank 2 anti-self-dual Yang-Mills equations by the introduction of Weyl co-ordinates (see, for example, Witten 1979). N.M.J.W. and L.J.M. (1988) showed that if these are solved locally by using the usual Ward transform, the holomorphic vector bundle over ordinary twistor space is actually the pull-back of a bundle E over a non-Hausdorff 'reduced' space.

Weyl co-ordinates are notorious for concealing the interesting parts of space-time geometry. For example, in the Schwarzschild solution, they only represent $R \geqslant 2m$, and the horizon is a part of the symmetry axis. (I shall use R to denote the radial co-ordinate in Kerr and Schwarzschild.) Since all the information about the analytic continuation of the manifold is contained in the exterior part of the metric, one might expect to find it in the twistor description; and this is in fact possible. The Kerr and Schwarzschild solutions have reduced twistor spaces which consist of two Riemann spheres S_0 and S_1 which are identified except at three pairs of points; these points are the points at infinity and $w = \pm(m^2-a^2)^{\frac{1}{2}}$ where w is a co-ordinate on the spheres and $a = 0$ in the Schwarzschild case. The bundle can be described in the standard form which consists of first restricting it to each sphere and then giving the patching matrix P between them. In each case, $E|_{S_0}$ is $L_1 \oplus L_0$; $E|_{S_1}$ is $L_{-1} \oplus L_0$; and

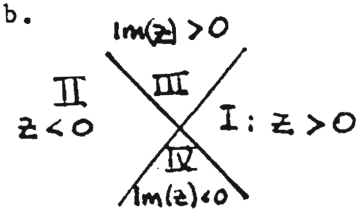
$$P = \frac{1}{w^2-m^2+a^2}\begin{pmatrix} (w+m)^2 + a^2 & 2am \\ 2am & (w-m)^2 + a^2 \end{pmatrix} .$$

(again, $a = 0$ for Schwarzschild). This description is unique if we demand that P be real (in the sense $\overline{P(w)} = P(\overline{w})$) and symmetric, and provided we know which points belong to which sphere in the reduced twistor space. We can, however, obtain a different patching matrix from the same bundle by changing our minds about which of the double points belongs to each of S_0 and S_1, and then putting the bundle in standard form over the two new spheres.

We therefore have four different possibilities. If we take S_0 and S_1 to be labelled by the points at infinity, then as well as the original description we can swap the points at +b, at −b or at both, where $b = (m^2-a^2)^{\frac{1}{2}}$. In order to see what this means in terms of the space-time, we have to introduce the idea of the patching matrix's being 'adapted' to one part of the axis in the (z,r)-plane. (Here z and r are the co-ordinates on the space of the orbits of the Killing vectors in the space-time; $r=0$ represents the symmetry axis.) This simply means that we can find the metric on the space of Killing vectors on this part of the axis by taking the limit as $r \to 0$ of its value in a neighbourhood of it. The patching matrix P given above is adapted to $z > b$, which corresponds to one half of the axis of symmetry in the space-time, outside the horizon. If we interchange the points at $w=b$, we get a matrix adapted to $-b<z<b$, which is the (outer) horizon; and if we interchange the points at both $w=b$ and $w=-b$, we have a patching matrix adapted to the other half of the axis, where $z<-b$.



In each case, for both Kerr and Schwarzschild, the bundle E restricted to S_0 is $L_1 \oplus L_0$ and the metric on the space of Killing vectors can be extended analytically to the axis or horizon. Moreover, in the region where $-b<z<b$, we can continue this metric to the region where it is negative definite. To do this, we use the same construction as before (see Woodhouse and Mason 1988), but take values of r which are purely imaginary. Thus an orbit of the Killing vectors, which is representd by the pair of points $w = z + ir$ and $w = z - ir$, now corresponds to a pair of points on the real axis in the reduced twistor space. By taking r to be both positive and negative (when it is real) and $\mathrm{Im}(r)$ to be both positive and negative (when r is imaginary), we can construct the usual cross-over at $R = m + b$.



We can now choose either to identify regions I and II, and regions III and IV, or to put in the orbit $(z,r) = (b,0)$ which corresponds to the cross-over itself. It can be shown that regularity of the metric (on the space of Killing vectors) at this point depends on the singularity structure of the patching matrix at $w=b$.

If we swap the points $w=-b$ in the Kerr solution, we find that we get another patching matrix Q of the kind that produces a cross-over (that is, Q is negative definite on the real axis in the w-plane); and that E restricted to S_0 is still $L_1 \oplus L_0$. If we take Q to be adapted to $-b<z<b$, then we can construct a similar picture to the one above; and the patching matrix adapted to $z>b$ turns out to be the inverse of the original matrix P. Since this can be obtained from P by replacing m with $-m$, the exterior region is now a negative mass Kerr solution. This of course must contain the ring singularity; the conjecture is that this is represented by the pull-back of E to the fibre of (Euclidean) twistor space above the appropriate points being non-trivial.

To obtain the Penrose diagram for the Kerr solution (see, for example, Hawking and Ellis 1973 p165) we have to identify region III for the '+b' crossover with region IV for the '-b' crossover. This can be done by considering the effect on the patching matrices of a reflexion of the (z,r) plane in the line $z = 0$. We also have to do the conformal rescaling which allows us to adjoin $\mathscr{I}$ to the space-time; it is at the moment less clear how the possibility of doing this is shown up by the twistor picture.

What is clear, however, is the difference between the Kerr and Schwarzschild solutions. For the latter, interchanging the points at $w=+m$ gives the same cross-over picture as for Kerr; but when we make the switch at $w=-m$, we find that the bundle E restricted to the new S_0 becomes $L_2 \oplus L_{-1}$. It is straightforward to see that this leads to a pole in the metric on the space of Killing vectors as $r \to 0$; this is of course the usual curvature singularity at $R = 0$.

<u>References</u>

S.W.Hawking & G.F.R.Ellis 'The large scale structure of space-time' C.U.P. 1973

L.Witten Phys. Rev. D <u>19</u> 718-20 (1979)

N.M.J.Woodhouse & L.J.Mason Nonlinearity <u>1</u> 73-114 (1988)

Many thanks to N.M.J.W.

James Fletcher.

A twistor transform for Hermitean symmetric spaces.

A familiar construction in twistor theory is the twistor transform which expresses the fact that a zero rest mass field can be given by a contour integral of a function of homogeneity $-n-2$ on twistor space $\mathbb{PT}^+$ or $+n-2$ on dual twistor space $\mathbb{PT}^{*-}$. Coupled with the conjugation $Z^\alpha \mapsto \bar{Z}_\alpha$ from $\mathbb{PT}^- \to \mathbb{PT}^{*-}$ and the pairing (called the dot product)

$$H^1(\overline{\mathbb{PT}^+}, \mathcal{O}(-n-2)) \times H^1(\overline{\mathbb{PT}^-}, \mathcal{O}(n+2)) \xrightarrow{\cdot} H^3(\mathbb{PT}, \Omega^3) \cong \mathbb{C}$$

(which is cup product to $\overline{\mathbb{PT}^+} \cap \overline{\mathbb{PT}^-} = \mathbb{PN}$ followed by Meyer-Vietoris connecting homomorphism) this gives a construction of certain unitary representations of $SU(2,2)$. The great advantage of this construction is that it is inherently geometrical (which allows one, for example, to define elementary states & calculate them using relative cohomology). In (1,2) MGE & I have extended twistor theory as an $SU(2,2)$ construction to arbitrary $\mathbb{C}$-semi-simple Lie groups. This note is a brief summary of what is known about possible generalizations of the twistor transform to other groups & spaces.

<u>An example</u>: In the usual twistor theory we use the Hermitean form $Z^\alpha \bar{Z}_\alpha$; consider instead $Sp(2,\mathbb{C})$ acting on $\mathbb{C}^4$ & preserving a symplectic form ω. In place of $SU(2,2)$ use $Sp(2,\mathbb{R})$. Then conjugation is $Z^\alpha \longrightarrow \bar{Z}^\alpha$ (conj. componentwise) & instead of $\mathbb{PT}^\pm$ etc we can consider $(\mathbb{CP}^3)^\pm = \{[V] \in \mathbb{CP}^3 \mid i\omega(V,\bar{V}) \gtrless 0\}$. The machine (Penrose Transform) of [1], [2] shows that

$$n \geq 0: \quad H^1(\overline{(\mathbb{CP}^3)^+}, \mathcal{O}(-3-n)) \cong H^1(\overline{(\mathbb{CP}^3)^+}, \mathcal{O}(-1+n)) \quad (*)$$

(a slight variation on the twistor case). Conjugation sends $\mathbb{CP}^{3+}$ to $\mathbb{CP}^{3-}$ (since ω is skew!) $\mathbb{CP}^3 = \times\!\Leftarrow\!\bullet$ (with a contact structure induced by ω); the role of minkowski space is played by $\bullet\!\Leftarrow\!\times = \mathbb{CS}^3$ (= 3 dim $\mathbb{C}$-minkowski space), regarded as a symplectic Grassmanian. [<u>Note</u>: $n=-1$: $(*)$ is obviously an isomorphism, but the cohomology is not irreducible].

<u>The general Hermitean symmetric case</u> Both standard twistor theory & the example are instances of a general construction.

To see this in terms of ordinary twistor theory recall that we can think of $\mathbb{M}^+$ in two ways: (i) as a complex manifold = open $SU(2,2)$ orbit in $\mathbb{M}$, (ii) as the real homogeneous space $SU(2,2)/S(U(2)\times U(2))$. In the latter guise it is not immediately clear (i) is true. We can ask, for general real semi-simple Lie groups G_0 with maximal compact subgroups K_0 when G_0/K_0 is a complex manifold, i.e. an open G_0 orbit in a complex homogeneous space G/P. ANSWER: see table below (for G_0 classical). The example just given is of this form. QUESTION: Is there a Twistor transform for such G_0/K_0? I.e. can we find $Q, Q' \subset G$ st. G/Q & G/Q' are complex manifolds with a twistor transform between them? ANSWER: Yes (see table) & what is more on either there is always a natural $\mathbb{Z}$'s worth of line bundles $\mathcal{O}(k)$, as on $\mathbb{CP}^3$ etc; as in example, it often happens that $Q = Q'$.

TABLE:

G_0	K_0	G/P	G/Q, G/Q'	Twistor transform: $n > 0$
$SU(p,q)$	$S(U(p)\times U(q))$		and	$H^{p-1}(\mathcal{O}(-p-n)) \cong H^{q-1}(\mathcal{O}(-q+n))$
$SO(2n+1,2)$	$U(1)\times SO(2n+1)$		(self dual)	$H^{p(p-1)/2}(\mathcal{O}(-2p+1-n)) \cong H^{p}(-1+n)$
$Sp(p,\mathbb{R})$	$U(p)$		(" ")	$H^{p-1}(\mathcal{O}(-p-1-n)) \cong H^{p-1}(-p+1+n)$
$SO(4p,2)$	$U(1)\times SO(4p)$		&	$H^{2p(2p-1)/2}(\mathcal{O}(-4p+2-n)) \cong H^{2p-1}(-2+n)$
$SO(4p+2,2)$	$U(1)\times SO(4p+2)$		(self dual)	$H^{(2p+1)p}(\mathcal{O}(-4p-n)) \cong H^{2p}(-2+n)$
$SO^*(2p)$	$U(n)$		(" ")	$H^{p-1}(-p-n) \cong H^{p-2}(-p+2+n)$

In all of these, there are two open G_0-orbits on G/Q (& Q') & the transform is between corresponding orbits. A relative cohomology form of elementary state exists & $G/Q \longrightarrow G/Q'$ under complex conjugation; so all the twistor arguments should go through to produce unitary representations of each G_0.

Thanks to MGE. & ED.

Rob Baston

[1] MGE/RJB TN 20.
[2] " "The Penrose Transform — its interaction with representation Theory" OUP. (1989).

29

A conformally invariant connection

Toby Bailey

November 4, 1988

This note is a postscript to the last section of my article in TN26 on the conformally invariant connection associated to a direct sum decomposition of one of the spin bundles. The general result lying behind the observations in that article , stated for convenince in the holomorphic category, is:

Theorem: *Let $\mathcal{M}$ be a complex conformal manifold with conformal metric g_{ab} and a given tensor field $J_a{}^b$ with $J_{(ab)} = 0$ and $J_a{}^c J_c{}^b = -\delta_a{}^b$. Then there exists a unique torsion-free connection ∇_a satisfying*

$$\nabla_a J_b{}^a = 0$$

$$\nabla_a g_{bc} = X_a g_{bc} \quad \text{for some} \quad X_a.$$

The second condition is simply that the conformal metric is preserved.

If one is given a direct sum decomposition of $\mathcal{O}^A$ in a complex space-time then $J_a{}^b = i(o_A \iota^B + \iota_A o^B)\epsilon_{A'}{}^{B'}$, where o_A, ι_A constitute a spin-frame defining the decomposition, satisfies the above conditions and the resulting connection is given in components by RPs 'conformally invriant edth and thorn' operators.

The significance of the rather strange condition on the derivitive of $J_a{}^b$ which defines the connection is unclear, and work continues on the use of this connection in type D conformal space-times and related areas.

Thanks to MGE and MAS.

30

Complex paraconformal manifolds—their differential geometry and twistor theory

T.N. Bailey*   M.G. Eastwood

October 7, 1988

Abstract

A complex paraconformal manifold is a pq-dimensional complex manifold ($p, q \geq 2$) whose tangent bundle factors as a tensor product of two bundles of ranks p and q. We also assume that we are given a fixed isomorphism of the highest exterior powers of the two bundles. Examples of such manifolds include 4 dimensional conformal manifolds (with spin structure) and complexified quaternionic, quaternionic Kähler and hyperKähler manifolds.

We develop the differential geometry of these structures, which is formally very similar to that of the special case of four dimensional conformal structures [30].

The examples have the property that they have a rich twistor theory, which we discuss in a unified way in the paraconformal category. In particular, we consider the 'non-linear graviton' construction [29], and discuss the structure on the twistor space corresponding to quaternionic Kähler and hyperKähler metrics.

We also define a family of special curves for these structures which in the 4-dimensional conformal case coincide with the conformal circles [34,2]. These curves have an intrinsic, naturally defined projective structure. In the particular case of complexified $4k$-dimensional quaternionic structures, we obtain a distinguished $8k + 1$ parameter family of special curves satisfying a third order ODE in local coordinates.

*This work was carried out with support from the Australian Research Council. T.N.B. would also like to thank the University of Adelaide for hospitality.

Preprints Available.

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Conformal Circles and Parametrizations of Curves in Conformal Manifolds

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July 26, 1988

Abstract

We give a simple O.D.E. for the conformal circles on a conformal manifold, which gives the curves together with a family of preferred parametrizations. These parametrizations endow each conformal circle with a projective structure. The equation splits into two pieces, one of which gives the conformal circles independent of any parametrization, and another which can be applied to *any* curve to generate explicitly the projective structure which it inherits from the ambient conformal structure [1].

We discuss briefly the use of conformal circles to give preferred co-ordinates and metrics in the neighbourhood of a point, and sketch the relationship with twistor theory in the case of dimension four.

*This work was carried out with support from the Australian Research Council. T.N.B. would like to thank the University of Adelaide for hospitality during this time.

AMS subject classifications. Primary 53A30; Secondary 58G30, 58G35.

Preprints Available.

32

On the Twistor Description of Sourced Fields

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12 October 1988

Abstract

Massless fields with source on an analytic world-line are double-valued, and it was shown by Bailey [1985] that a large family of such fields have a twistor description in terms of relative cohomology groups. In this paper it is proved that *all* right-handed massless fields are obtained in this way, and that if the sheaves $\mathcal{O}(n-2)$ are quotiented by the polynomials, then the relative cohomology of the resulting sheaves describes all left-handed sourced massless fields. The proof for right-handed fields uses techniques developed by Singer [1987,1988] for applying the Penrose transform to situations in which the 'pull-back mechanism' is non-trivial. For the left-handed fields it is necessary to use some additional arguments involving the conserved quantities (e. g. momentum and angular momentum for spin 2) of these fields; it is shown that the conserved quantities are the obstructions to a twistor description of left-handed fields in terms of the cohomology of $\mathcal{O}(n-2)$.

Relative cohomology and projective twistor diagrams

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10 October 1988

Abstract

The use of relative cohomology in the investigation of functionals on tensor products of twistor cohomology groups is considered and yields a significant reduction in the problem of looking for contours for the evaluation of (projective) twistor diagrams. The method is applied to some simple twistor diagrams and is used to show that the standard twistor kernel for the first order massless scalar ϕ^4 vertex admits a (cohomological) contour for only one of the physical channels. A new kernel is constructed for the ϕ^4 vertex which admits contours for all channels.

33

A Hamiltonian Interpretation of Penrose's Quasi-Local Mass

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Abstract

A connection is established between Penrose's definitition of quasi-local mass and the more conventional notions of mass and momentum etc. arising from the canonical formalism of general relativity (which exist at least asymptotically). It is shown that the each component of the 'angular momentum' twistor can be thought of as the value of a Hamiltonian which generate motions of regions of the space-time which tend towards one of a collection of 'quasi-Killing vectors' on the bounding 2-surface on which the computations take place. The quasi-Killing vectors are obtained from solutions of the twistor equation, and essential use is made of the spinorial version of the gravitational Hamiltonian first employed in Witten's simplified proof of positive energy in general relativity.

These ideas are then used to suggest a variation on Penrose's quasi-local mass definition using 'quasi conformal Killing vectors' rather than quasi-Killing vectors. This has the advantage that there are only 16 *real* quantities rather than the 20 real (10 complex) ones from Penrose's original definition.

[†]Esmée Fairbairn Junior Research Fellow and Andrew Mellon Postdoctoral Fellow supported also in part by NSF grant no. PHY 80023.+ FULBRIGHT SCHOLARSHIP.

To appear in Classical and Quantum Gravity

Insights from Twistor Theory

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Abstract

This article discusses how twistor methods may be applied to problems arising from the canonical quantization of gravity. First of all the hypersurface twistor space construction is briefly reviewed, and a correspondence between (complex) initial data sets and a complex 3-manifold together with two cohomology classes is described.

Three possible applications of the methods are discussed. Firstly, a polarization condition analagous to that of positive frequency for initial data sets is presented. Secondly, it is argued that a canonical quantization procedure based on the use of the twistorial data would realize Penrose's suggestion that one should quantize gravity in such a way as to 'fuzz' out space-time points, leaving null directions well defined; the usual procedure smears the metric and therefore the null directions but leaves the space-time events fixed. Thirdly, it is pointed out that the gauge group for the twistor data is unrelated to the space-time diffeomorphism group so that the technical difficulties associated with factoring out the diffeomorphism group can be avoided.

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To appear in "Conceptual Problems in Quantum Gravity", Osgood Hill Conference Proceedings, Boston.

35

Bäcklund transformations for the anti-self-dual

Yang-Mills equations

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Beginning from any given (local) solution of the GL(n,$\mathbb{C}$) anti-self-dual Yang-Mills (ASDYM) equations on Minkowski space, a simple technique for the generation of large classes of solutions (perhaps in some sense all) is given. The origin of this technique is described in terms of two versions of the Ward construction. The resulting description of Bäcklund transformations is sufficiently simple that it is then possible to identify the group generated by the collection of all such Bäcklund transformations and the space on which it acts in terms of concrete functions.

in J. Math. Phys. 29 (4) April 1988

Abstracts:-

Poon's Self-Dual Metrics
and Kähler Geometry.

Claude LeBrun

It is shown that the self-dual conformal metrics on connected sums of $\mathbb{CP}_2$'s recently produced by Y. S. Poon arise from zero scalar curvature Kähler metrics on blow-ups of $\mathbb{C}^2$ by adding a point at infinity and reversing the orientation.

[In J Differential Geometry 28 (1988) 341-343]

The Integral Constraint Vectors of Traschen
and Three-Surface Twistors.

K. P. Tod.

I relate the integral constraint vectors (ICV's) of Traschen to covariant constant sections of a connection at a hypersurface. I find the conditions for ten linearly independent ICV's to exist as the vanishing of the curvature of this connection. I interpret these conditions in terms of embeddability of the hypersurface in a space of constant curvature and I relate the ICV's to 3-surface twistors.

[In Gen Rel and Grav, 20 № 12, 1988]

Twistor Newsletter No. 27

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