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Cohomological Residues

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This is a corrected and much improved version of an old TN article [1]. There are three sections. In §1 we generalise the dot product slightly and relate it to the connecting homomorphism α derived from the short exact sequence

$$0 \longrightarrow \mathcal{O}(-n-1) \overset{xs}{\longrightarrow} \mathcal{O}(-n) \longrightarrow \mathcal{O}_{\{s=0\}}(-n) \longrightarrow 0 \ . \qquad (1)$$

Then, in §2, we show how to describe Leray's residue map in terms of cohomology. We conclude by demonstrating a long-suspected relationship between "cohomological evaluation" and the residue map.

1. *The connecting map α and the dot product*

The sheaves in (1) are defined on the open subset X of $\mathbf{CP}^n$. s is a holomorphic function on X (such as $A_\alpha X^\alpha$) and we let S be the zero set of s in X and U be an open neighbourhood of S in X. Now we have the two maps

$$H^q(U;\mathcal{O}(-n)) \overset{r}{\rightarrow} H^q(S;\mathcal{O}(-n) \overset{\alpha}{\rightarrow} H^{q+1}(X;\mathcal{O}(-n-1)) \qquad (2)$$

where r is simply restriction and α is the connecting homomorphism of (1). Let $\omega\epsilon H^q(U;\mathcal{O}(-n))$. We claim that

$$\alpha(r(\omega)) = \omega \cdot \frac{1}{s} . \tag{3}$$

Proof: We start by describing $\alpha(r(\omega))$ in terms of Čech cohomology. Let $\{\Sigma_a\}$ be an open covering of U, and $\{X_i\}$ be an open covering of X-S. Suppose

$$\omega = \{\omega_{a_0 \ldots a_q}\} .$$

To obtain $r(\omega)$ we simply regard the $\omega_{a_0 \ldots a_q}$ as having been restricted to S. The map α is in three pieces. (i) We construct a q-cochain $\tilde{\omega}$ in X (with respect to the covering $\{\Sigma\} \cup \{X\}$) as follows:

$$\tilde{\omega}_{a_0 \ldots a_q} = \omega_{a_0 \ldots a_q}$$

but $\tilde{\omega} \ldots = 0$ if the (q+1)-fold intersection includes any sets from $\{X_i\}$.
(ii) We take the coboundary of $\tilde{\omega}$:

$$(\delta\tilde{\omega})_{a_0 \ldots a_{q+1}} = 0 \quad \text{because } \omega \text{ was a cocycle,}$$

$$(\delta\tilde{\omega})_{i a_0 \ldots a_q} = \tilde{\omega}_{a_0 \ldots a_q} = \omega_{a_0 \ldots a_q} ,$$

$$(\delta\tilde{\omega}) \ldots = 0 \quad \text{whenever the (q+2)-fold intersection contains more than one set from } \{X_i\}.$$

(iii) We divide by s. This makes sense because $\delta\tilde{\omega}$ puts $\omega_{a_0 \ldots a_q}$ on all sets $X_i \cap \Sigma_{a_0 \ldots a_q}$ and zero on all other (q+2)-fold intersections. But this is exactly the Čech definition of

$$\omega \cdot \frac{1}{s} ,$$

where s is thought of as an element of $H^0(\{X_i\};\mathcal{O}(-1))$.

2. *The residue map*

If φ is a holomorphic form closed on X-S and with a pole of order 1 on S then Leray's residue theorem [2] says that there exist forms ψ and θ such that

$$\varphi = \frac{ds}{s} \wedge \psi + \theta \tag{4}$$

where $\psi|_S$ is closed and holomorphic. $\psi|_S$ is called res(φ). In terms of mappings between cohomology groups this residue map comes in two parts.
(i) We think of φ as an element of $H^0(X-S;\Omega^p)$ and then we use the relative cohomology exact sequence

$$H^0(X-S;\Omega^p) \xrightarrow{c} H^1(X,X-S;\Omega^p) \to H^1(X;\Omega^p) \tag{5}$$

to map φ to a pair (ω,η) (representing $c(\varphi)$), where

$$\omega \in \Omega^{p,1}(X), \qquad \eta \in \Omega^{p,0}(X-S),$$

$$\bar\partial\omega = 0, \quad \bar\partial\eta = \omega|_{X-S} \; .$$

Here $(\omega,\eta) \sim (0,\varphi) \sim (\bar\partial\beta\wedge\varphi,\beta\varphi)$ where β is any C^∞ bump function identically 1 on S and with support in an arbitrary neighbourhood of S. (See [3] for the Dolbeault description of relative cohomology). (ii) We contract the normal bundle of S in X to a disc bundle

$$D \xrightarrow{\pi} S \; .$$

Then we squeeze the bump β until (ω,η) is supported in D. Finally we integrate ω along the fibres of π (i.e. over the discs) to get

$$\pi_*(\omega) \in \Omega^{p-1,0}(S) \;.$$

In fact this induces a map between the cohomology groups

$$\pi_*: H^1(X,X-S;\Omega^p) \to H^0(S;\Omega_S^{p-1}) \;.$$

It can be seen that if φ were of the form (4) then $\pi_*(c(\varphi)) = \psi|_S$, as required.

We can generalise the maps c and π_* (and specialise p to n = dimX) to obtain the following commutative diagram (in which the top row is exact).

$$\begin{array}{ccccc} H^q(X-S;\Omega^n) & \xrightarrow{c} & H^{q+1}(X,X-S;\Omega^n) & \xrightarrow{\text{forget}} & H^{q+1}(X;\Omega^n) \\ & \searrow^{\text{res}} & \downarrow{\scriptstyle \pi_*} & \nearrow_{\alpha} & \\ & & H^q(S;\Omega_S^{n-1}) & & \end{array}$$

(Here we have also used the facts that on X we have $\Omega^n = \mathcal{O}(-n-1)$ while on S we have $\Omega_S{}^{n-1} = \mathcal{O}_S(-n)$). In particular, therefore, $\alpha_0 \text{res} = 0$. This result doesn't quite capture the folklore relationship between the dot product and the residue map, however. So we start again.

3. *Cohomological Evaluation and the Residue Map*

Consider

$$\omega \in H^0(X-S_1 \cup S_2 \cup \ldots \cup S_m; \Omega^p) .$$

Since $X-\cup S_j$ is covered by $U_j = X - S_j$ there are various interpretations of ω by various Čech (Mayer-Vietoris) maps. The simplest case is when $m = 2$. Now the Čech map is

$$H^0(X-S_1-S_2;\Omega^p) \to H^1(X-S_1 \cap S_2;\Omega^p) \tag{6}$$

and the question (posed in section 2 of [4]) is: which contours in $H_p(X-S_1-S_2)$ "factor through" this interpretation? The answer (see [4]) is to consider the dual Mayer-Vietoris sequence

$$\ldots \longrightarrow H_{p+1}(X-S_1 \cap S_2) \xrightarrow{\partial_*} H_p(X-S_1-S_2) \longrightarrow \ldots \tag{7}$$

and look for contours in the image of this map.

In Dolbeault terms, the map (6) is

$$\omega \longrightarrow \omega \wedge \bar{\partial}\beta$$

where $\beta \in C^\infty(X-S_1 \cap S_2)$ and

$$\beta = \begin{cases} 0 & \text{near } S_1 \\ 1 & \text{near } S_2 \end{cases} .$$

All this is well known (and described in [4]). What was not known was its intimate relation to the taking of residues. We use the characterisation that

$$\int_{\delta\gamma} \varphi = \int_{\gamma} \mathrm{res}(\varphi)$$

(where δ is the cobord map) and the following remarkable connection between Leray's exact sequence and Mayer-Vietoris.

Lemma

Consider the two Leray sequences

$$\begin{array}{ccccccc}
\cdots \longrightarrow & H_{p+1}(X-S_1 \cap S_2) & \overset{\cap_a}{\longrightarrow} & H_{p-1}(S_1-S_2) & \overset{\delta_a}{\longrightarrow} & H_p(X-S_1) & \longrightarrow \cdots \\
& \uparrow & & \| & & \uparrow & \\
\cdots \longrightarrow & H_{p+1}(X-S_2) & \overset{\cap_b}{\longrightarrow} & H_{p-1}(S_1-S_2) & \overset{\delta_b}{\longrightarrow} & H_p(X-S_1-S_2) & \longrightarrow \cdots
\end{array}$$

Then the composite $\delta_b \cap_a$ is equal to the Mayer-Vietoris connecting homomorphism ∂_* in (7).

Proof

We use a description in terms of compactly supported differential forms, whereby classes in $H_k(M)$ are represented by closed elements of

$\Omega_c^{\dim M-k}(M)$ (see [3] for details).

In these terms,

$$\partial_*(\alpha) = \alpha \wedge d\beta$$

where β is C^∞ on $X - S_1 \cap S_2$ and

$$\beta = \begin{cases} 0 & \text{near } S_1 - S_1 \cap S_2 \\ 1 & \text{near } S_2 - S_1 \cap S_2 \end{cases}$$

To describe $\cap_a$, let $j: S_1 - S_2 \to X - S_1 \cap S_2$ be the inclusion; then $\cap_a(\alpha) = j^*(\alpha)$. The description of δ_b is a little more involved.

Let D be a tubular neighbourhood of $S_1 - S_2$ relatively compact in $X - S_2$, with projection map π. Let β be C^∞ on $X - S_1 \cap S_2$, chosen so that

$$\beta = \begin{cases} 0 & \text{near } S_1 - S_1 \cap S_2 \\ 1 & \text{in a neighbourhood of } X - D \end{cases}$$

This is a specialisation of our earlier definition.

If $[\chi] \in H_{p-1}(S_1 - S_2)$, the form $\pi^*(\chi) \wedge d\beta$ represents $\delta_b[\chi]$.

The composite thus carries α to $\pi^* j^*(\alpha) \wedge d\beta$. Because $j_0\pi$: D→D is homotopic to the identity map, there exists an operator H such that

$$\pi^* j^* u - u = dHu + Hdu$$

for all forms u in D. Applying this to α, we find

$$\pi^* j^*(\alpha)\wedge d\beta = -\alpha\wedge d\beta = d(H\alpha\wedge d\beta)$$

and since $H\alpha\wedge d\beta$ has compact support in $D-S_1$, we see that $\pi^* j^*(\alpha)\wedge d\beta$ and $\alpha\wedge d\beta$ represent the same class in $H_p(X-S_1-S_2)$.

Comments

1. Note that S_2 could be replaced by a union of closed submanifolds $S_2,\ldots,S_m$ without any change. It is, however, essential that S_1 be a (single) closed submanifold.

2. The promised intimacy between res and ∂_* is given by the formula

$$\int_\kappa \omega = \int_{\cap_a \lambda} \mathrm{res}(\omega)$$

where $\lambda \in H_{p+1}(X-S_1\cap S_2)$ and $\kappa = \partial_*\lambda = \delta_b\cap_a\lambda$. Note that from the commutative diagram of the lemma, we have $\kappa = \partial_*\lambda$ if κ is in the image of δ_b and the image of κ in $H_p(X-S_1)$ is zero.

3. This explicit characterisation of which contours are 'cohomological' appears to be new, although widely guessed at. While suggestive, it stops short of being a complete account of the treatment of twistor diagram ears. Work is in progress.

References

1. SAH in TN 9.

2. Leray, J (1959) Bull. Soc. Math. France 87 81-180.

3. SAH and MAS "Relative Cohomology and Projective Twistor Diagrams" preprint.

4. SAH and MAS in TN 23.

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Abstract

Almost Hermitian Symmetric Manifolds I Local Twistor Theory

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Abstract

Conformal and projective structures are examples of structures on a manifold which are modelled on the structure groups of Hermitian symmetric spaces. We show that each such structure has associated a distinguished vector bundle (or *local twistor bundle*) equipped with a connection (*local twistor transport*). For projective and conformal manifolds, this is Cartan's connection. The curvature of the connection provides an tensor invariant which vanishes if and only if the manifold is locally isomorphic to a Hermitian symmetric space.