## Non-Hausdorff Riemann surfaces and complex dynamical systems

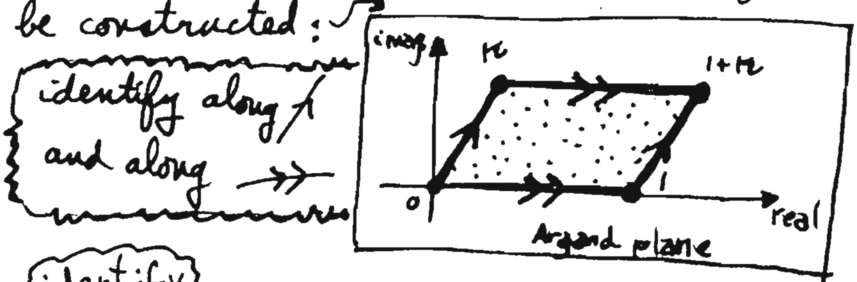
If twistor theory is ever to give a description of such "non-integrable" systems as the general Yang-Mills or Einstein equations, it must be able to accommodate chaotic behaviour (such as occurs in the Belinskii-Lifshitz-Khalatnikov or Misner discussion of the Bianchi type 9 vacuums). A simple type of chaotic system (~Julia sets; the Mandelbrot set) is given by iterated complex maps

$$z \mapsto z^2 + c \qquad (z, c \in \mathbb{C},\ c \text{ const}).$$

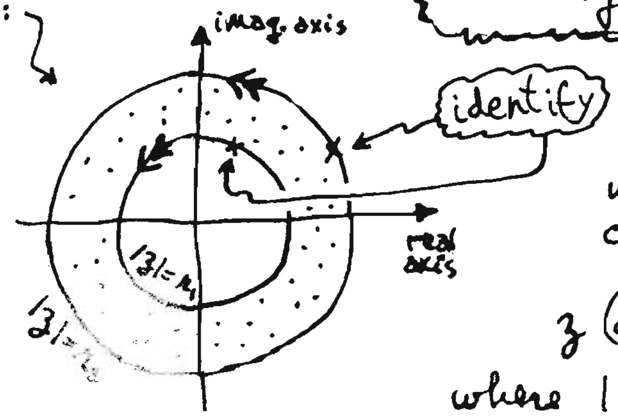
It turns out that such maps have a close relationship with a certain type of non-Hausdorff Riemann surface.

It should be noted that non-Hausdorff Riemann surfaces have already found significant application in the Woodhouse-Mason description of Ward's construction for stationary axi-symmetric solutions of Einstein's vacuum equations, so it is quite possible that non-Hausdorff surfaces of the type that I am considering may also find some significant role within twistor theory.

Let us first recall how an ordinary Riemann surface of genus 1 (torus) may be constructed:




An alternative procedure would be:




where the strip $r_1 \leq |z| \leq r_2$ is curled into a torus by identifying $z$ (at $|z| = r_1$) with $\lambda z$ (at $|z| = r_2$), where $|\lambda| = r_2/r_1$. Thus, in a certain sense, this Riemann surface codes the information of the map

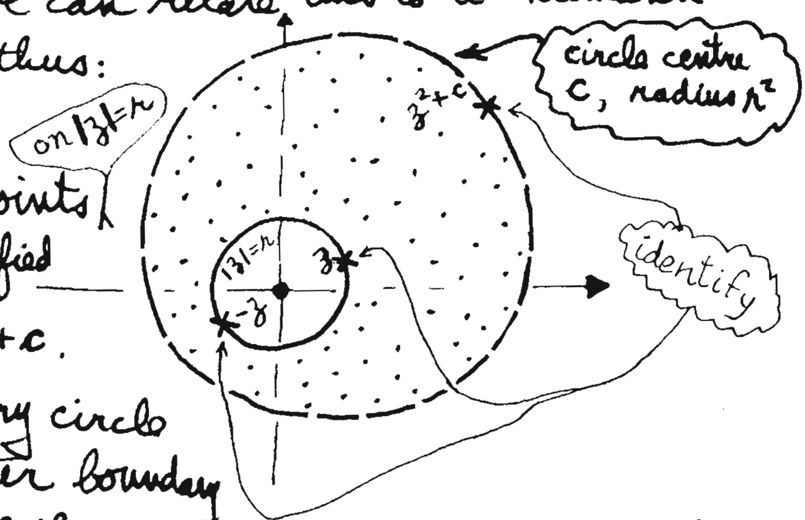
$$z \mapsto \lambda z ,$$

where we go once around  , and iterations of this map correspond to going many times around.

Now consider the quadratic map

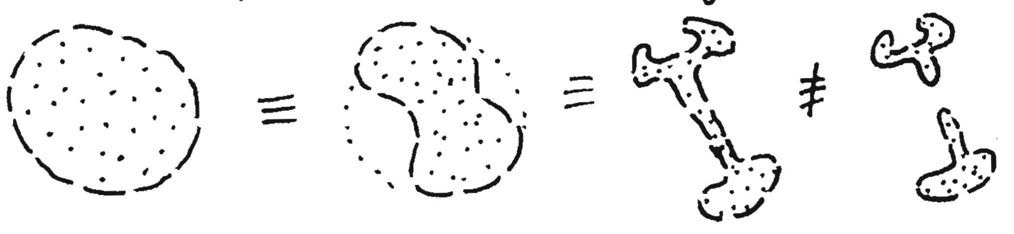
$$z \mapsto z^2 + c .$$

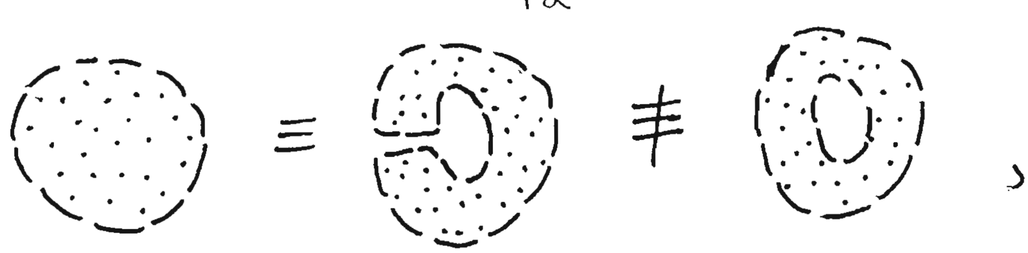
Similarly to the above, we can relate this to a "Riemann surface" S, constructed thus:




Here the pair of opposite points $z$ and $-z$ are to be identified with the single point $z^2+c$. Taking the inner boundary circle to be present and the outer boundary circle to be absent, we have constructed a non-Hausdorff Riemann surface. (Each point has a neighbourhood that is a small complex open disc, but on $|z|=r$, the discs for $z$ and for $-z$ always overlap.)

However, this singles out $|z|=r$ as "special", which we do not want, so we must adopt a viewpoint that removes this feature. It has already been apparent that some new concept of "complex manifold" is needed in twistor theory, in which we have an equivalence (for non-compact complex manifolds)

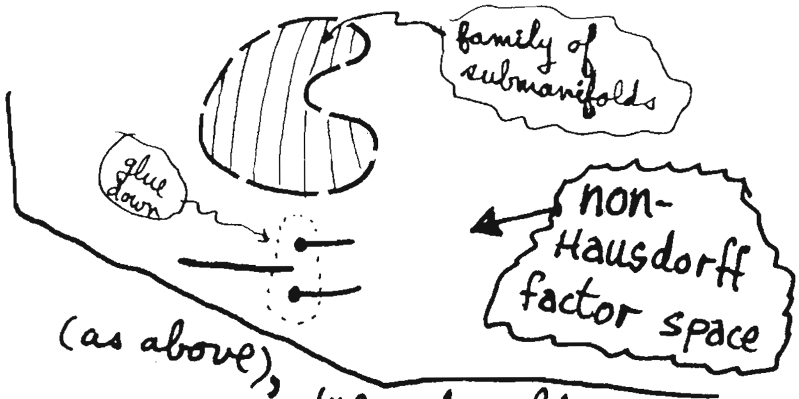
 ,


, etc.

Thus, the space can be "pared" away or built up at its (non-compact) edge, but not so as to give rise to a change in its global structure. The concept remains somewhat vague, as of now, but something of this sort is needed if we require elements of

$$H^1(X, \Theta) \quad \text{holomorphic vector fields}$$

to represent the infinitesimal deformations of a non-compact complex manifold $X$. (Deformations of the "boundary" don't show up in $H^1(X,\Theta)$.) Now consider how a non-Hausdorff complex manifold can often arise (as indeed is the case in the Woodhouse-Mason construction):




If we have:

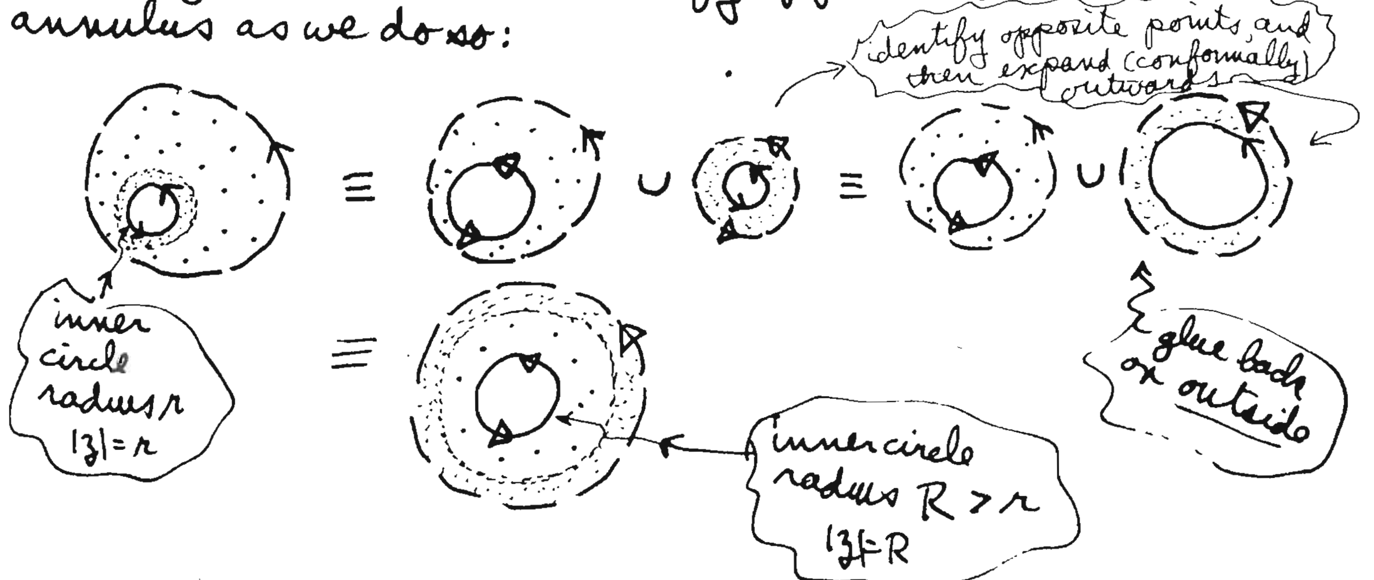

(as above), we should also have:


; and also


(since

). Thus we can "split" or "reglue" non-Hausdorffness, provided that (in some appropriate sense) we do not alter the "global connectivity" of the space. (I think that this can be stated in terms of what local deformations are allowed and what are not — but this needs more understanding.)

In the case of the non-Hausdorff Riemann surface just constructed, we can remove a small annulus from the inner circle boundary and glue it on at the outer circle boundary, but we must identify opposite points of this annulus as we do so:




Also, we can perform this operation in reverse so long as there remains room on the inside.

If $c$ is in the Mandelbrot set $M$, then the topological structure of $S$ (under this equivalence) appears to differ in an essential way from when $c$ is not in $M$. For in some sense $S$ has a "covering space" which is the complement $K$ of the Julia set in $\mathbb{C}$, and whether $c$ is in $M$ or not depends on the multiple-connectivity properties of $K$. (When $c \notin M$, $K$ has a Cantor set removed from it, but when $c \in M$, $K$ is topologically an annulus.)

Some clarifying ideas seem to be needed.

Roger Penrose

Advertisement:

"The Emperor's New Mind", by R. Penrose, is to be published by the O.U.P. in September 1989, if all goes according to schedule. (Twistor Theory is mentioned in two footnotes!)