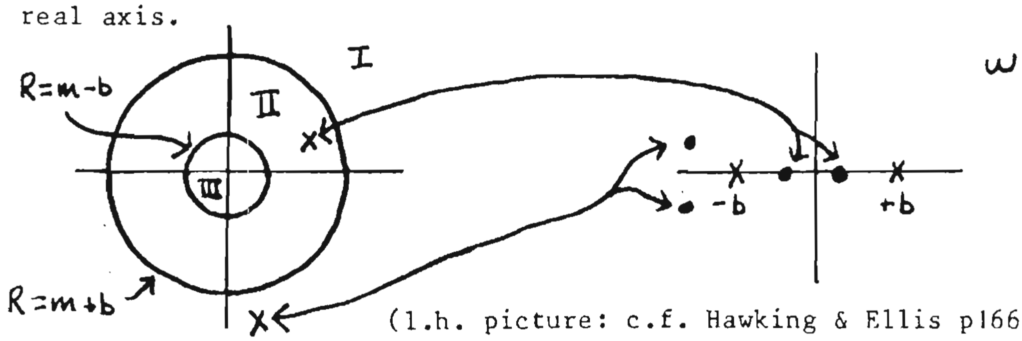
MORE ON THE TWISTOR DESCRIPTION OF THE KERR SOLUTION

In my article in TN 27, I outlined the relationship between the non-Hausdorff twistor spaces arising from NMJW's and LJM's construction and the geometry of the Kerr and Schwarzschild solutions. The purpose of this note is to expand one or two points that arose there.

Recall that the space of orbits of the two Killing vectors in the Kerr solution corresponds to the space of quadratic maps $p: X \to R_U$, where X is a copy of $\mathbb{CP}^1$ (with coordinate q) and R_U is the reduced twistor space consisting of two Riemann spheres (coordinate w) which are identified everywhere except for the pairs of points at infinity and at $w = \pm b$. In order to determine a map p, we need to know first the values of w for which the discriminant of the equation $w = p(q)$ vanishes, and then which point of X is mapped to each of the pair of points at both $w = +b$ and at $w = -b$. If we write p in the form

$$p(q) = \tfrac{1}{2}r(q^{-1} - q) + z,$$

then the two values of w are $z + ir$ and $z - ir$, where z and r are the usual Weyl coordinates; and p is determined by these and the choice of one of four possible treatments of the double points. Orbits of the Killing vectors outside the outer horizon or inside the inner one are given by real z and real, positive r, and therefore correspond to pairs of complex conjugates in the w-plane. Orbits between the two, on the other hand, have z again real but r purely imaginary; moreover z and r are constrained so that the points $z \pm ir$ lie between +b and -b on the real axis.

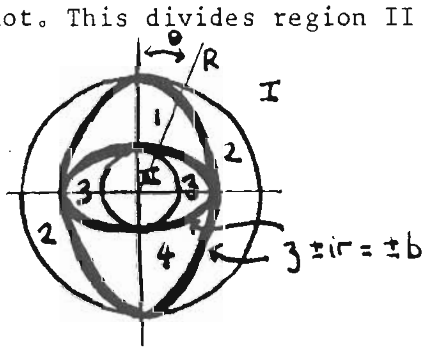


(l.h. picture: c.f. Hawking & Ellis p166)

There are, however, some values of z and r for which we cannot evaluate the metric directly by following the Ward splitting procedure. As NMJW and LJM showed in their paper, the method works provided the points $w = z + ir$ and $w = z - ir$ are distinct, and are both places where the two w-spheres are identified.

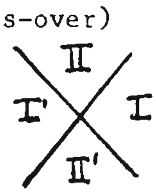
In the ordinary outside region I, therefore, the only problems can occur as $r \to 0$; and in their paper NMJW and LJM found the conditions on the bundle over R_U for the metric to be well-behaved on the axis of the Weyl coordinates (which corresponds to either an axis or a horizon in the space-time). Similarly, in region III, where the manifold can be continued analyticly out to $\mathcal{I}$, there are the two parts of the axis and the horizon, and, in addition, the ring singularity. In my previous article, I mentioned the conjecture that this might correspond to a map p for which the pull-back of the bundle over R_U to one over X is non-trivial. This does in fact turn out to be the case, and it can be shown that when $z = 0$ and $r = a$, the pulled-back bundle is $L_2 \oplus L_{-2}$.

By contrast, in region II we can have values of z and r such that one of the pair $z \pm ir$ coincides with one of $\pm b$, but the other one does not. This divides region II into four, as follows:



(volumes 2 and 3 are of course connected since there is rotational symmetry about the z-axis.)

and these four volumes correspond precisely to the four different maps p that exist for each pair (z,r), and thus to the four different possible treatments of the double points $\pm b$. This means that each pair of points on the real axis in the w-plane between +b and -b represents four orbits of the two Killing vectors in the space-time; and if we consider the analytic continuation of the space-time (putting in the point at the $R = m + b$ cross-over)



where regions I' and II' are isometric to I and II in the usual way, we actually have eight orbits for each pair. (In I and II, we define r by $r = -\frac{1}{2}i(w_1 - w_2)$; in I' and II', we take $r = +\frac{1}{2}i(w_1 - w_2)$.)

This raises various questions. If each pair of points (w_1, w_2) on the real axis between +b and -b corresponds to four orbits in the space-time, why is the same not also true of each pair of complex conjugates, or even for each general pair of points, in the w-plane? Secondly, how can we tell that the space-time is in fact regular across the hypersurfaces where one of the points coincides with +b or -b; and what do these surfaces mean geometrically?

The answer to the first question is that in the complexification of the Kerr solution each (ordered) pair of points does represent four Killing vector orbits, but not all of these intersect the real slice which is the space-time. Thus there are four real orbits for pairs in (-b,+b) on the real axis, two real orbits (one in region I and one in region III) for pairs of distinct complex conjugates, and none otherwise. Trying to find another real orbit for the pairs of conjugates would be equivalent to interpreting the axis in volume 1 of region II as a horizon; and this would be incompatible with regularity at the orbit $w_1 = +b = w_2$. Outside the outer horizon we are forced to think of $r = 0$ as an axis since it is the space-like Killing vector which vanishes on it. If $r = 0$ were a horizon then J, the metric on the space of orbits, would change signature to (+,+) across it.

We can also use the analyticity of the complexification to see that J is well-behaved across the boundaries between volumes 1,2,3 and 4. By considering small variations of z into the complex, we can move (z,r) from one volume to another without $z \pm ir$ coinciding with $\pm b$, but with the explicit effect of changing the treatment of the double points by the corresponding maps $p:X \to R_U$. This is made clear by the behaviour of the open sets covering X on whose overlaps the pull-back of the bundle over R_U is described by the pull-backs of the patching matrices in the standard form I described before.

Finally, the boundaries themselves in fact represent the light-cones of the two points where the axis and the (outer) cross-over intersect.

References

NMJW and LJM Nonlinearity 1 73-114 (1988)

JF 'Non-Hausdorff Twistor Spaces for Kerr and Schwarzschild' TN 27

SW Hawking & GFR Ellis 'The Large Scale Structure of Space-time' CUP 1973

Thanks to NMJW.

James Fletcher.