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On Ward's Integral Formula for the Wave Equation in Plane Wave Space-Times

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In [1], Ward presents an integral formula for the general solution of the wave equation in plane wave space-times. The purpose of this note is to show how this relates to the twistor integral formula in flat space, and to generalize the formula to arbitrary helicity. The generalization of the formula shows how it is that Maxwell theory satisfies a kind of Huygens principle in plane wave space-times. This suggests further generalizations of Huygens principle. However, electromagnetic fields in plane wave space-times provide the only nontrivial example of such generalized Huygen's principles.

Ward's integral formula:
Consider the plane wave space-time with metric:

$$ds^2 = du \cdot dv - G_{ij}(v)dx^i \cdot dx^j \qquad i,j=1,2.$$

For convenience we shall choose a conformal scale such that $det(G_{ij})=1$ (all our considerations will be conformally invariant, so this doesnt involve any loss of generality). The hypersurfaces of constant u are null and support δ-function solutions, $\delta(u)$, of the wave equation, (this follows from $\Box u = 0 = (\nabla_a u)(\nabla^a u)$).

The hypersurfaces of constant $u_b = u + 2x^i b_i + F^{ij} b_i b_j$ (where $F^{ij} = \int G^{ij}(v)dv$) are related to the hypersurfaces of constant u by the symmetries

$$(u,\ v,\ x^i) \rightarrow (u + 2x^i b_i + F^{ij} b_i b_j,\ v,\ x^i + F^{ij} b_j),$$

and so also support δ-function solutions of the wave equation. We can form the general solution of the wave equation by averaging over these δ-function solutions:

$$\phi(x) = \int \Phi(u_b, b_i) d^2 b$$

where Φ is an arbitrary function of its three arguments. When $G_{ij}(v) = \delta_{ij}$, (flat space) the formula reduces to the Whittaker integral formula.

Relationship with the twistor integral formula
In flat space, this formula can be seen to be the twistor integral formula as follows. Write:

$$\pi_{1'}/\pi_{0'} = \zeta = b_1 + ib_2, \qquad \bar{\pi}_1/\bar{\pi}_0 = \bar{\zeta} = b_1 - ib_2.$$

Then $u_b = x^{AA'}\pi_{A'}\bar{\pi}_A / \pi_{0'}\bar{\pi}_0 = \omega^A \bar{\pi}_A / \pi_{0'}\bar{\pi}_0$ and we can put:

$$\Phi d^2 b \equiv (\bar{\pi}_0 \pi_{0'})^{-2} \Phi(\omega^A \bar{\pi}_A, \pi_{A'}, \bar{\pi}_A) \bar{\pi}^A d\bar{\pi}_A \wedge \pi^{A'} d\pi_{A'}$$

so that $(\bar{\pi}_0 \pi_{0'})^{-2} \Phi(\omega^A \bar{\pi}_A, \pi_{A'}, \bar{\pi}_A) \bar{\pi}^A d\bar{\pi}_A$ is a homogeneity degree -2 Dolbeault representative on twistor space constructed from the characteristic data at $\mathcal{I}$, Φ, for the field ϕ as in my TN article [2].

A difficulty with the twistorial interpretation of this formula in the curved case is that the appropriate complex structure on the space of primed spinors (on which b_i are coordinates) shifts as v varies; $\pi_{1'}(v)/\pi_{0'} = \zeta(v) = \bar{m}^i(v) b_i$, the b_i are held constant. The complex structure is determined by the 2-metric $G_{ij}(v) = \bar{m}_{(i} m_{j)}$. It is therefore not clear how one can obtain a global holomorphic interpretation of the formula in the conformally curved case. (One can, of course, provide a holomorphic interpretation of the formula on each of the hypersurface twistor spaces based on

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hypersurfaces of constant v; the above formula then answers, to a certain extent the question of how to identify cohomology classes based on one hypersurface with those on subsequent hypersurfaces.)

Generalization to higher helicity

Plane waves, and in fact all Brinkman waves, have a covariantly constant spinor, o^A, aligned along the generators of the hypersurfaces of constant v. This can be used to raise and lower helicity of massless fields.

If $\phi(x)$ is a solution of the wave equation, then $\varphi_{AB}=o^{A'}\nabla_{AA'}o^{B'}\nabla_{BB'}\phi(x)$ is an ASD solution of the Maxwell equations. All solutions of the Maxwell equations can be put in this form (this follows from $[o^{A'}\nabla_{AA'},o^{B'}\nabla_{BB'}]=0$ together with $o^{A'}\nabla^A_{A'}\varphi_{AB}=0$ from Maxwell's equations). Similarly, all solutions of the neutrino equations can be put in the form $o^{A'}\nabla_{AA'}\phi(x)$. Higher helicity fields constructed in this way will not, in general, satisfy the Z.R.M. equations because of Buchdahl conditions. However, there is a consistent potentials modulo gauge description

$$\psi_{AA'_1\cdots A'_{n-1}}=o_{A'_1}\cdots o_{A'_{n-1}}o^{B'}\nabla_{AB'}\phi(x)$$

satisfies the $(n-1)$-potential equations.

This description leads to the following formula for the general solution of Maxwell's equations:

$$\varphi_{AB}(x)=\int o^{A'}\nabla_{AA'}\, o^{B'}\nabla_{BB'}\Phi(u_b,b_i)d^2b.$$

Note that the first ∇ in this expression acts on the free spinor index on the second ∇ as a *covariant* derivative. Let $\partial_{AA'}$ denote the coordinate derivative in the spinframe determined by the null tetrad $l=dv$, $n=du$, $m=m_i dx^i$, $\bar{m}=\bar{m}_i dx^i$ where $m_{(i}(v)\bar{m}_{j)}(v)=G_{ij}(v)$ and the phase of m_i is determined by the condition that $\dot{m}_{[i}\bar{m}_{j]}+\dot{\bar{m}}_{[i}m_{j]}=0$ (the dot, $\cdot$, denotes $\partial/\partial v$). (Note that this last condition together with $\det(G_{ij})=1$ implies that $\dot{m}_i=\bar{\sigma}\bar{m}_i$ for some $\bar{\sigma}(v)$.) Then the spin coefficients are just $\gamma_{aB}{}^C=\sigma\iota_{A'}o_A o_B o^C$ and $\psi_4=\dot{\sigma}$. This formula then becomes, using coordinate derivatives in the above spin frame:

$$\varphi_{AB}(x)=\int\left\{o^{A'}\partial_{AA'}\, o^{B'}\partial_{BB'}\Phi(u_b,b_i)+\sigma o_A o_B(\partial_{u_b}\Phi)\right\}d^2b.$$

For higher helicity, the 'field' versions of these formulae fail to make reasonable sense because of Buchdahl conditions, however the potentials modulo gauge formulae do make sense.

[In the flat case, $\sigma=0$, write $\bar{\pi}_A=\bar{\pi}_0 o^{A'}\partial_{AA'}u_b$ and $\bar{\alpha}=(\bar{\pi}_0)^{-4}(\partial^2_{u_b}\Phi)\left(\overline{(\bar{\pi}_0)^{-2}\bar{\pi}^A d\bar{\pi}_A}\right)$. Then the above becomes the Dolbeault version of the (-4)-homogeneity complex conjugate (dual) twistor integral formula

$$\varphi_{AB}(x)=\int\bar{\pi}_A\bar{\pi}_B\bar{\alpha}\wedge\bar{\pi}_A d\bar{\pi}^A.$$

If we put $\frac{\partial}{\partial \omega^A}=(\pi_{0'})^{-1}o^{A'}\partial_{AA'}$, the formula generalizes the 0-homogeneity twistor integral formula.]

Huygen's principle

In Penrose (1972) it is demonstrated that a δ-function solution of the Maxwell's equation in a conformally curved plane wave space-time must also have a 'tail'. One can produce such solutions easily using the above ideas. Pick a null hypersurface, $u=0$. Let $\theta(u)$ be the Heavyside function, $\theta(u)=1$ for $u>0$, and $\theta(u)=0$ for $u<0$. Since any function of u is a solution to the Laplacian, we have in particular that $\phi=u\theta(u)$ is a solution. This means that $A_{AA'}=o_{A'}o^{B'}\nabla_{AB'}\phi=o_{A'}\iota_A\theta(u)$ is a vector potential solution to Maxwell's equations. The corresponding field is

$$\varphi_{AB}=\iota_A\iota_B\delta(u)+o_Ao_B\sigma\theta(u)$$

which has the tail $o_Ao_B\sigma\theta(u)$.

If, instead, we start with $\phi=\theta(u)$ we obtain a δ-function vector potential, $A_{AA'}=o_{A'}\iota_A\delta(u)$, without a tail, and so the field, $\varphi_{AB}=\iota_A\iota_B\delta'(u)+o_Ao_B\sigma\delta(u)$, also has no tail. This would seem to suggest a generalized form of Huygen's principle in which it is sufficient to have solutions supported on light cones or (as we have shown above) solutions supported on a family of null hypersurfaces such that there is at least one hypersurface in the family normal to each null direction through each point. The relevant solutions may be a sum of the first n derivatives of δ-functions thus leading to a hierarchy of Huygen's principles, $\mathcal{H}^n$, in which for the n^{th}, only the first n derivatives of δ-functions are allowed.

An alternative formulation of Huygen's principle is that if one poses initial data on some hypersurface Σ, then the solution at a point p depends only on the data at the intersection of the light cone L_p of p with Σ. The generalizations would then seem to correspond to requiring that the solution at p depends only on the data at $L_p\cap\Sigma$ and its first n derivatives. It would be interesting to find an example of $\mathcal{H}^\infty$.

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Ward R S (1987) Progressing waves in flat space-time and in plane-wave space-times, CQG 4, 775.

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