Hypersurface Twistors

The purpose of this note is to describe some results concerning hypersurface twistors and initial data. The first results describe the extent to which hypersurface twistor spaces can be used to encode initial data. In particular I discuss how the situation changes significantly when the hypersurface is chosen to be a light cone or at infinity, where, roughly speaking, half the data is lost. This fact leads to some of the difficulties one has with the 'Googly'. I also discuss (the failure of) a twistorial definition of positive frequency for initial data sets on hypersurfaces.

Encoding initial data
Before one can start using and applying hypersurface twistors, there arises the question of how they can be used to encode initial data sets for the Einstein vacuum equations. This question can be divided into two parts. The first part is how the conformally invariant part of the initial data set can be encoded (this consists of the conformal structure on a three manifold and the trace free part of the extrinsic curvature). The second part is concerned with encoding the conformal factor of the three metric and its time derivative, the trace of the extrinsic curvature.

There are (at least) two approaches to encoding initial data into hypersurface twistor spaces. I refer to one approach as the 'real' approach, and the other as the 'complex' approach. Let Σ be the complexification of a real space-like hypersurface $\Sigma_{\mathbb{R}}$ in an analytic Lorentzian space-time M. Let $P\mathcal{T}$ be the hypersurface twistor space, and PN be the (real) codimension one hypersurface in $P\mathcal{T}$ whose points correspond to hypersurface twistors which intersect $\Sigma_{\mathbb{R}}$. In the real approach one is allowed to know the location of the hypersurface PN in $P\mathcal{T}$ whereas in the complex approach one is only allowed $P\mathcal{T}$ as a complex manifold and holomorphic structures thereon.

The real approach: This is now relatively well understood, Sparling (1983) LeBrun (TN9,1984&1985), Penrose (1984), Mason (1985). The real approach is relatively easy to compute with as calculations can all be performed locally on PN using the Chern-Moser connection. LeBrun showed that PN as a CR manifold determines $\Sigma_{\mathbb{R}}$ and the conformally invariant part of the initial data. In Mason (1985) it was also shown that one could encode the information of the conformal factor if one introduced a homogeneity degree two (1,0)-form, ι, which generalizes $I_{\alpha\beta}Z^{\alpha}dZ^{\beta}$ from flat space twistor theory. The constraint equations could then be articulated as $I_{[\alpha\beta;\gamma]}=0$. In order to obtain a formula for the evolution, it was necessary to introduce a further homogeneity degree two (1,0)-form, σ, which generalizes $H_{\alpha\beta}Z^{\alpha}dZ^{\beta}$; σ encodes the information of the location of the hypersurface Σ in M^4, in flat space a point $X^{\alpha\beta}$ corresponds to a point of Σ iff $X^{\alpha\beta}H_{\alpha\beta}=0$. This approach has various defects; from a twistorial point of view, the data depends on free functions of 5 variables, as compared to 3-variables for the gravitational field initial data. The characterization of those CR manifolds corresponding to gravitational initial data sets requires the knowledge of the location of the $\mathbb{CP}^1$'s in PN correponding to points of $\Sigma_{\mathbb{R}}$. These are then determined locally using the Chern Moser connection. These facts substantially limit the applicability of this approach.

The complex approach: This has not been much studied, but has presumably been around in folklore for some time. The basic observation is that, using a generalization of the nonlinear graviton construction, $P\mathcal{T}$ can be seen to be the twistor space of a 4-dimensional conformal manifold, $\mathcal{M}^4$, with ASD Weyl curvature into which Σ^3 is embedded together with its conformal equivalence class of initial data (conformal 3-metric and trace free part of the extrinsic curvature), LeBrun (1979). The space $\mathcal{M}^4$ is colloquially known as 'heaven on earth'. In order to capture the conformal initial data set, one must encode the information of the location of Σ^3 in $\mathcal{M}^4$.

When the extrinsic curvature is pure trace, the location of Σ can be encoded by means of a global holomorphic homogeneity degree 2 1-form, σ, on $P\mathcal{T}$ (σ vanishes on restriction to those holomorphic curves corresponding to points of Σ). When the extrinsic curvature is general it appears to be impossible to encode the location of Σ in $\mathcal{M}$ using local holomorphic structures on $P\mathcal{T}$; the σ as defined in the 'real approach' is no longer holomorphic. However, it is straightforward to encode the location of Σ using a cohomology class which, abusing notation, we can also call σ. The cohomology class σ can be taken, for instance, to be the element of $H^1(P\mathcal{T},\mathcal{O}(-2))$ corresponding to the solution of the wave equation on $\mathcal{M}^4$ which is zero on Σ and whose normal derivative is some lapse function. As a consequence we have that for space-like hypersurfaces the conformal equivalence class of initial data is encoded in $P\mathcal{T}$ together with σ. The conformal factor can be similarly encoded by means of the cohomology class $\iota\in H^1(P\mathcal{T},\mathcal{O}(-2))$ which corresponds to the solution of the wave equation which is 1 in the desired conformal scale.

The holomorphic approach has the important advantage that the twistor data consists of effectively free functions of three variables. I have not as yet been able to articulate the constraint and evolution equations in this context. Insight into this problem would perhaps be obtained from relating the real and complex approaches; the real and complex approaches should be related in much the same way as Dolbeault is related to Čech cohomology. However, one may need to use more sophisticated cohomology classes for ι and σ in the holomorphic approach such as elements of $H^1(P\mathcal{T},\Omega^1(2))$.

Light cones and $\mathscr{I}$: The canonical hypersurface twistor spaces are those where the hypersurface is taken to be one of past or future null infinity. If the above results were to hold for $\mathscr{I}$, then the structures σ and ι would coincide, thus reducing the complexity of the description. Another way to reduce the complexity of the description is to use light cones as initial data hypersurfaces since then the information of the location of the hypersurface is encoded simply as the quadric, $Q\subset P\mathcal{T}$, whose points correspond to the generators of the null cone, $\mathcal{N}$. (Often, when defining hypersurface twistor spaces for null hypersurfaces, Q is deleted from $P\mathcal{T}$. It can be checked that Q embeds holomorphically in $P\mathcal{T}$. I am including Q since deleting it only reduces the amount of information available.) There are three cases to consider; null $\mathscr{I}$, space-like $\mathscr{I}$ and a light cone $\mathcal{N}$. Unfortunately, in all these cases half the initial data is lost. (This is of particular irritation when one hopes to use asymptotic twistor space as the basic twistor space for the googly construction.)

Space-like $\mathcal{I}$: The hypersurface twistor space construction encodes, as usual, the intrinsic conformal structure of $\mathcal{I}$ and the (trace free part of) the extrinsic curvature which vanishes. However the free asymptotic data consists of the intrinsic conformal structure of $\mathcal{I}$ and its *third* derivative into the space-time (the first derivative of the electric part of the Weyl curvature at $\mathcal{I}$). We therefore see that 'half' the data, the third derivative of the conformal structure, is lost.

'Heaven in church': This is the colloquial name for the heaven construction based on a light cone. As in the other heaven constructions, one obtains a space-time $\mathcal{M}^4$ with ASD Weyl curvature into which the hypersurface $\mathcal{N}$ is embedded. The 'heaven', $\mathcal{M}^4$, is constructed as the space of holomorphic curves with S^2 topology in $P\mathcal{T}$ and $\mathcal{N}$ consists of those curves which intersect the quadric Q in $P\mathcal{T}$. The hypersurface $\mathcal{N}$ acquires initial data from its embedding in $\mathcal{M}^4$. However, this initial data is not the original set. In order to see this, consider the case where the quadric, Q, in $P\mathcal{T}$ can be blown down to a line L in some complex manifold $\widetilde{P\mathcal{T}}$ with $P\mathcal{T}\setminus Q\simeq\widetilde{P\mathcal{T}}\setminus L$. This then implies that the 'heaven in church' construction embeds the null hypersurface, $\mathcal{N}$, as a light cone in $\mathcal{M}^4$ as $\widetilde{P\mathcal{T}}$ can be taken to be the twistor space for $\mathcal{M}$ and L can be taken to be the curve in $\widetilde{P\mathcal{T}}$ corresponding to the vertex of the light cone. This means that, according to the induced initial data from $\mathcal{M}$, $\mathcal{N}$ is foliated by α-planes and therefore the $\sim$'ed shear vanishes. We therefore see that, roughly speaking, half the data on $\mathcal{N}$ is lost. C. LeBrun has shown that it is always possible to blow down Q when $P\mathcal{T}$ is close to the hypersurface twistor space of a null cone in $\mathbb{M}$; the existence of a regular blowdown only requires conditions on the normal bundle of Q. In order to encode the extra data, one needs to also have the 'time' rate of change of the complex structure($\in H^1(P\mathcal{T},\Theta)$) as the hypersurface is evolved through the space-time. (This data will, of course, be subject to constraints.)

Null infinity: Null infinity suffers from the combination of the two above difficulties. Not only does the hypersurface twistor space fail to encode half the data, but also the second half of the data only appears as a holomorphic vector valued (0,1)-form, $\ddot{\bar\partial}$, which is the *second* derivative of the $\bar\partial$ operator as the hypersurface is evolved to second order into the space-time. This can be computed as follows. In the space-time with unphysical metric in which $\mathcal{I}$ is a finite null hypersurface with normal $\iota^{A'}\iota^{A}$, one can compute the evolution of the $\bar\partial$ operator to be:

$$\dot{\bar\partial}=\mathcal{L}_{\frac{z^{A'}\bar{z}^{A}}{|z\cdot\iota|^{-2}}\nabla_{AA'}}\bar\partial=\frac{\bar{z}_{A}\iota_{A'}dx^{AA'}}{(z\cdot\iota)^2(\bar{z}\cdot\bar{\iota})}\Psi_{B'C'D'E'}z^{B'}z^{C'}z^{D'}\frac{\partial}{\partial z_{E'}}$$

Here the hypersurface twistor space at $\mathcal{I}$ is coordinatized by the coordinates, x^a of $\mathcal{I}$ itself and the spinor valued coordinate $z_{A'}$ up the fibre of the spinor bundle restricted to $\mathscr{I}$. Since Ψ vanishes on $\mathscr{I}$, $\dot{\bar{\partial}}$ vanishes and $\ddot{\bar{\partial}}$ has the identical form to the above expression with Ψ replaced by its first derivative, Ψ^0, in the direction transverse to $\mathscr{I}$ ($\nabla_{AA'}\Psi_{B'C'D'E'} \dot{=} \iota_A \iota_{A'} \Psi^0_{B'C'D'E'}$). This expression is somewhat messy to write out in terms of the asymptotic shears and their integrals (it depends on integrals of non-linear combinations of σ and $\bar{\sigma}$). However, when $\bar{\sigma}=0$, (the 'googly' case) the hypersurface twistor space at $\mathscr{I}$ is ordinary flat twistor space, and $\ddot{\bar{\partial}}$ should be simple. However I havent yet worked it out as the details turn out to be slightly more problematic than expected. The cohomology class defined by $\ddot{\bar{\partial}}$ should vanish since hypersurface twistor spaces in self-dual space-times based on null cones are all flat. We should therefore have that $\ddot{\bar{\partial}} = \bar{\partial} V$ for some (1,0) vector field V.

A definition of positive frequency for gravitational initial data sets

One of the more striking results in twistor theory is the geometrization of the positive/negative frequency splitting for ZRM fields that is so important in quantum field theory. This might lead one to suggest (Mason 1989) that an initial data set should be said to be of positive frequency if the correponding hypersurface twistor space could be continued from a neighbourhood of PN with topology $S^3\times\mathbb{R}\times S^2$ to a region with topology $\mathbb{R}^4\times S^2$ which would be thought of as a deformed analogue of $\mathbb{PT}^+$ (so that one can fill in the $S^3\times\mathbb{R}$ factor into a ball$\simeq\mathbb{R}^4$). (Cf the definition of a positive frequency non-linear graviton in Penrose 1976.)

One can check what this definition does in linearized theory by taking the expression for infinitesimal deformation of a hypersurface twistor space in flat space, $\mathbb{PT}$, due to a linearized solution of the field equations that I obtained in my D.Phil. thesis (see also my article in TN 20). The condition would require that the deformation be of positive frequency. This implies that the contribution from the ASD part of the field is of positive frequency since that appears directly in the formula. However, the expression for the infinitesimal deformation uses the reflection of the SD part of the field in the hypersurface. This reflection must be of positive frequency, so that the SD part of the field must be of negative frequency. As Abhay Ashtekar has pointed out, this is unfortunately unphysical, since it implies that the helicity of both the ASD and the SD contributions both have negative helicity, so that one has the tensor product of two $-$ve helicity graviton Fock spaces in linearized theory with this definition, instead of the product of the positive with the negative.

This definition may have some interest, since, firstly there are many Lorentzian real initial data sets which can satisfy this condition (at least in linearized theory), and for such initial data sets one will have a canonical twistorial definition of positive frequency for backround coupled ZRM fields.

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