Some Stein Complementary Series for $GL(4,\mathbb{C})$

We are all familiar with the massless fields on $\mathbb{M}$, $\mathbb{M}^{\pm}$ and $\mathbb{M}^{0}$. On $\mathbb{M}^{\pm}$ they turn-out to be singular unitary representations of the group $U(2,2)$ (or $SU(2,2)$, if you prefer). Here the word 'singular' can be interpretted in a variety of ways. One version of the technical definition is to say that they do not contribute to the Plancherel measure for G. That is, when decomposing $L^2(G)$ (in analogy to the Fourier transform for $L^2(\mathbb{R})$) the singular representations don't appear. Alternatively, these are the representations which (almost) everybody missed when first trying to classify unitary representations.

The group $GL(4,\mathbb{C})$ has singular unitary representations of a type slightly different from the massless fields. These were first recognized by E. Stein. His lectures at Univ. of Notre Dame on his results make good reading. To write down these representations, proceed as follows:

As a homogeneous space for $G = GL(4,\mathbb{C})$, $\mathbb{M} = G/P$ where
$$P = \left(\begin{array}{c|c} L_1 & B \\ \hline 0 & L_2 \end{array}\right).$$

Here, L_1, L_2, B and 0 are 2×2 blocks. The isotropy group P has a decomposition $P = MAN$ where

$$M = \left\{ \left(\begin{array}{c|c} L_1^0 & 0 \\ \hline 0 & L_2^0 \end{array}\right) \;\middle|\; |\det L_1^0| = 1 \text{ and } |\det L_2^0| = 1 \right\}$$

$$A = \left\{ \left(\begin{array}{c|c} \ell_1 I & \\ \hline & \ell_2 I \end{array}\right) \;\middle|\; \ell_1 \text{ and } \ell_2 \text{ are positive real \#'s} \right\}.$$

$$N = \left\{ \left(\begin{array}{c|c} I & B \\ \hline 0 & I \end{array}\right) \right\} \qquad (I = 2\times 2 \text{ identity matrix.})$$

Thus, N is unipotent, A is abelian and M and A commute with each other. An irreducible representation of P is naturally a tensor product of irreducibles for M and A and the trivial representation of N. Via the standard principal bundle theory, each representation of P induces a homogeneous vector bundle on G/P. For the complementary series line bundles will suffice. The one-dimensional representations of P are its characters. The unitary characters of M are

$$\xi(m_1, m_2)\left(\begin{array}{c|c} L_1^0 & \\ \hline & L_2^0 \end{array}\right) = (\det L_1^0)^{m_1} (\det L_2^0)^{m_2}$$

where m_1 and m_2 are integers. Since A is abelian, all its irreducibles are one-dimensional:

$$\nu(s_1, s_2)\left(\begin{array}{c|c} l_1 I & \\ \hline & l_2 I \end{array}\right) = l_1^{s_1} l_2^{s_2} \qquad s_j = \sigma_j + i\tau_j .$$

These are unitary when $\sigma_1 = \sigma_2 = 0$. The resulting line bundle on $\mathbb{M}$ carries an invariant Hermitian inner product when ν is unitary. The representation of G on the L^2-sections is a 'unitarily induced' representation. They are well-known and not singular. However, the parameters s_j can move off the imaginary axis a little and still produce unitary representations. In the present case, "a little" means:

$$-2 < \sigma_1 < 2$$
$$-2 < \sigma_2 < 2 .$$

Now, however, the invariant inner product is gone since the inner product on the fibers is not invariant when $\sigma_j \neq 0$. To get the right Hermitian pairing, one constructs intertwining operators between the representation $\pi_{\xi,\nu}$ and its Hermitian dual. The Hermitian dual of a representation π on a Hilbert space is defined by

$$\pi^h(g) = \pi(g^{-1})^*$$

where $*$ means take the adjoint operator. To define the adjoint here, use the non-invariant inner product. Using an analytic continuation argument starting from the $\sigma_j = 0$ cases, it is possible to show that the intertwining operators, $A(s_1, s_2)$, are bounded, self-adjoint and invertible until one encounters poles at $\sigma_1 = \pm 2$, $\sigma_2 = \pm 2$. The operators for unitary ν are bounded, self-adjoint and invertible to begin with, since the representations of G are unitary, hence equivalent to their duals. The intertwining operators give new invariant inner products on the $\pi_{\xi,\nu}$ by defining

$$\langle \varphi, \psi \rangle^A = \langle \varphi, A\psi \rangle .$$

When the parameters s_1, s_2 reach the poles, the inner product will be either undefined or not positive definite.

Remarks:

① I don't know what these representations look like when restricted to $U(2,2)$. They will generally be reducible. For instance, they should contain copies of

$$\Gamma(\mathbb{M}^{\pm}, \mathcal{O}[m_1][m_2]').$$

One hope would be that they contain a sequence of interesting representations in some fashion similar to the way massless field representations come from restricting the metaplectic representation to $U(2,2)$.

② When forming the line bundle on $\mathbb{M}$ one gets a holomorphic bundle only when the starting representation is holomorphic. The representations used here look like:

$$\begin{pmatrix} L_1 & \\ & L_2 \end{pmatrix} \longmapsto \frac{(\det L_1)^{m_1}(\det L_2)^{m_2}}{|\det L_1|^{s_1-m_1}\,|\det L_2|^{s_2-m_2}},$$

which is not holomorphic for $s_1 \neq m_1$, $s_2 \neq m_2$.

Ed Dunne.

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Nonlinear Schrödinger and Korteweg-de Vries are Reductions of Self-Dual Yang-Mills

L.J.Mason[†‡] & G.A.J. Sparling[‡], University of Pittsburgh.

Abstract

The non-linear Schrödinger (NS) and KdV equations are shown to be reductions of the self-dual Yang-Mills (SDYM) equations. A correspondence between solutions of the NS and KdV equations and certain holomorphic vector bundles on a complex line bundle over the Riemann sphere is derived from Ward's SDYM twistor correspondence. Remarkably the twistor correspondence generalizes to the NS and KdV hierarchies when complex line bundles of higher Chern class are used. We discuss solitons and inverse scattering.

†Andrew Mellon postdoctoral Fellow and Fulbright Scholar. Present address: The Mathematical Institute, 24-29 St Giles, Oxford OX1 3LB, England.

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