## Double box diagrams

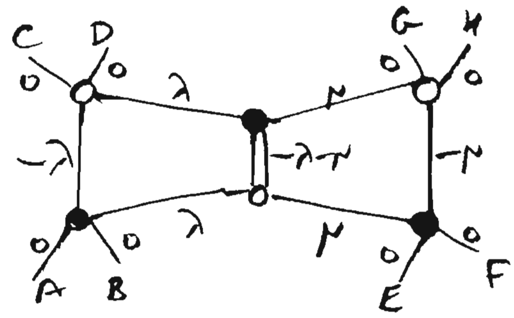
**Introduction**: The attempt [TN 27] to perform the direct translation of Feynman propagators and vertices in general position led to the writing down of large twistor diagrams too difficult to evaluate at present. These large diagrams essentially arise as compositions of simpler "box" diagrams. It therefore looks useful to approach general diagram-bulding by a detailed study of the simplest such composition, namely the double box. There are several other reasons for studying it:

(1) the early study of the double box by RP and George Sparling [PhD thesis, 1974] has remained uncompleted.

(2) it turns out that new light is cast on the crossing symmetry problem for the *single* box.

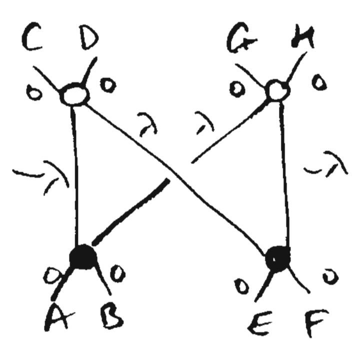
(3) there are applications to electromagnetic and SU(2) amplitudes of particular interest.

**The problem: Sparling's approach.** In what follows we shall use only the original projective diagram calculus. To begin with we can confine ourselves to *scalar* (elementary) external states, so that the most general double box diagram we need consider is

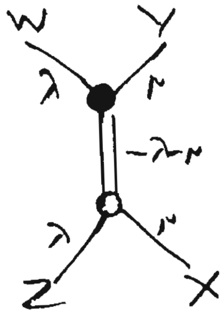



Here $\lambda$ and $\mu \in \mathbf{C}$ cannot be non-zero integers; (but we shall be particularly interested in studying limits as they approach integer values.)

We now follow Sparling's program for an explicit evaluation. The idea of this approach is that we first integrate out $U_\alpha$ and $V^\alpha$, reducing the integral to one which we can treat by the methods used for the single box diagram



To do this we first note that there is a contour for




allowing $Z^\alpha = X^\alpha$, $W_\alpha = Y_\alpha$, the result of the integral being

$$\Gamma(1+\lambda)\Gamma(1+\mu)(W.Z)^{-1-\lambda}(Y.X)^{-\mu-1}\, I_p\left(\lambda+1, \mu+1; \frac{W.Z\ Y.X}{Y.Z\ W.X}\right)$$

where $I_p(\lambda+1, \mu+1; u)$

is the hypergeometric function defined by $\int_0^\infty dz\, z^\mu (1+zu^{-1})^{-1-\lambda}(1+z)^{-1-\mu}$

Sparling left the problem at essentially this point. To pursue it to a conclusion we have to analyse the "period" contours for the UV integral, i.e. contours corresponding to the period structure of the hypergeometric function. The key point is that only a very particular choice of contour will yield a result which is actually an ***amplitude***: i.e. a mapping from in-states and out-states to **C**. The choice is such that in the result of the UV integral,

$I_p(u)$ is replaced by $I_H(u) = \{(1-e^{2\pi i\lambda})+(1-e^{2\pi i\mu})\} I_p(u) + I_c(u)$

where $I_c(u)$ is $\oint_{\text{round } [-1,0]} dz\, z^\mu (1+zu^{-1})^{-1-\lambda}(1+z)^{-1-\mu}$

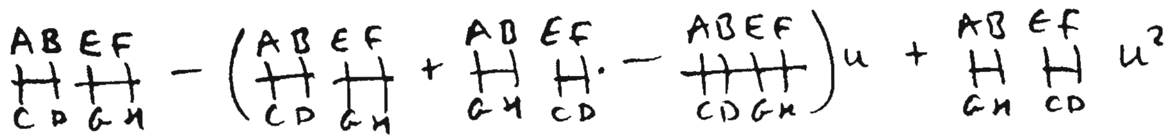
The defining feature of this peculiar combination is that the *period* of $I_H(u)$

about the branch point at $u = 0$ is $(1-e^{2\pi i\lambda})(1-e^{2\pi i\mu})\, I_p(u)$

Explicit contours can be constructed for the WZ, then the YX integrals, essentially in analogy with the integration of the single box, although this requires care. The final result is, in closed form

$$\frac{I_p(\lambda,\mu; r_1) - I_p(\lambda,\mu; r_2)}{r_1 - r_2} = \frac{1}{2\pi i}\oint \frac{I_p(\lambda,\mu; u)}{Q(u)}\, du$$

where Q(u) is the standard quadratic



formed from the four external states, and $r_1, r_2$ are its roots. Note that

$$I_P(\lambda,\mu;u)$$

is analytic at u = 1. This is the feature which ensures that we now have a genuine amplitude for AB, EF corresponding to in-states; CD, GH to out-states. Had we chosen another contour at the first stage we should still obtain a finite integral (at least for AB ... GH in general position) but without this essential physical property.

The result can be rewritten as the non-compact integral $\displaystyle\int_0^\infty \frac{I_C(\lambda,\mu;u)\,du}{Q(u)}$

where $\displaystyle I_C(\lambda,\mu;u) = \oint_{\substack{\text{round}\\ [-1,\,0]}} dz\, z^{\mu-1}(1+zu^{-1})^{-\lambda}(1+z)^{-\mu}$

the representation being valid provided $\mathrm{Re}\,\lambda > -1, \quad \mathrm{Re}\,\mu > -1$

in analogy with $\displaystyle\int_0^\infty \frac{u^\lambda\,du}{Q(u)} \qquad (-1 < \mathrm{Re}\,\lambda < 1)$

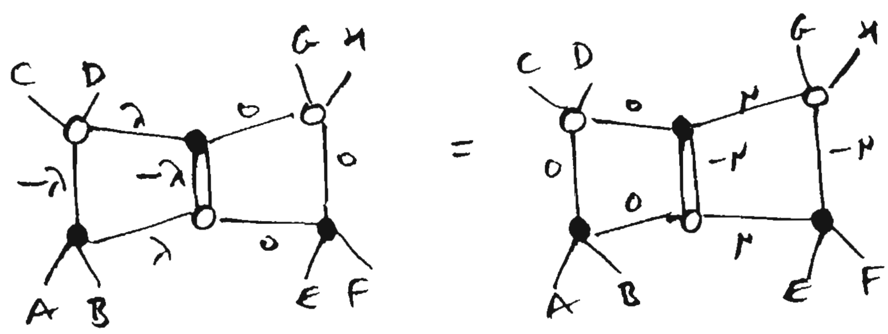
for the single box.

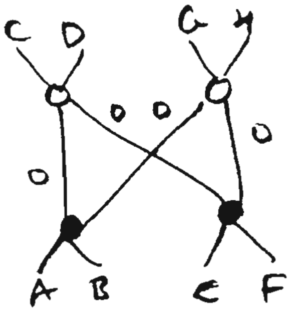
**Deductions from the solution:**

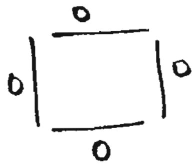
This allows us to note at once some special cases:

(i): $\lambda = 0$ or $\mu = 0 \implies I_C(\lambda,\mu;u) \equiv 1$

hence



$$= \int_0^\infty \frac{du}{Q(u)} =$$ 


This is consistent with 

acting as a two-twistor projection operator for eigenstates of spin 0.

(ii) considering the limits

$$\mu \to 1: \quad I_c(\lambda, \mu; u) \to (1-u^{-1})^{-\lambda}$$
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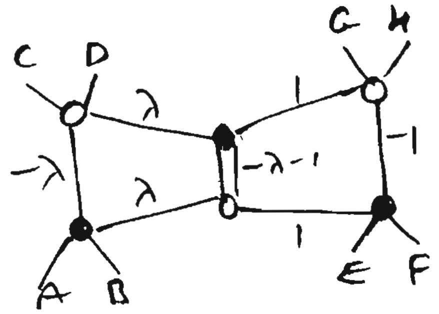
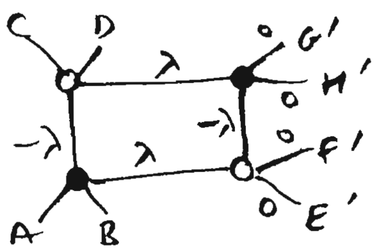
we find



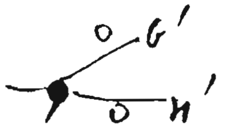

$$= \int_0^\infty \frac{(1-u^{-1})^{-\lambda}}{Q(u)}\, du$$

which one may check is just the amplitude previously obtained by taking a so-called "hard contour" for




$$(GH = G'H', \quad EF = E'F')$$

This means that the *double* box (with $\mu = 1$) supplies a genuine contour integral involving genuine $H^1$s for the "missing" channel of the *single* box. Note that the "hard contour" so-miscalled, is one which treats two of the external $H^1$s as if they were $H^0$s. Such a contour can better be described by writing the $H^0$s as

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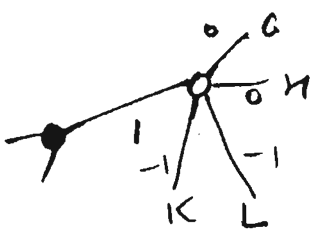
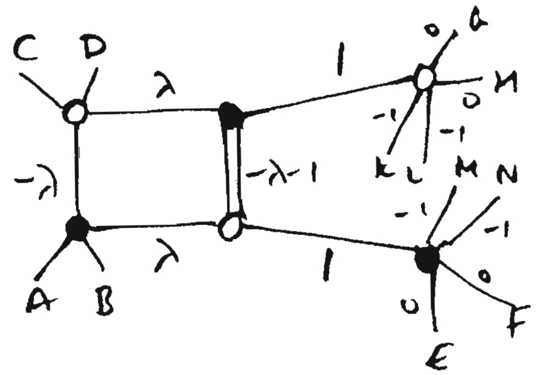
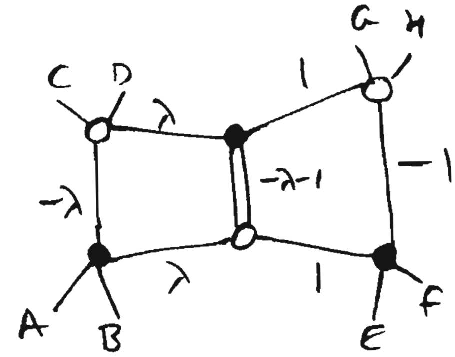


so that the external state is genuinely an $H^1$ and parameter-dependence is explicitly given by the K, L . The old results on "hard contour" integration (APH *Physica* **114A**, 157 (1982)) can now be recast as the statement that

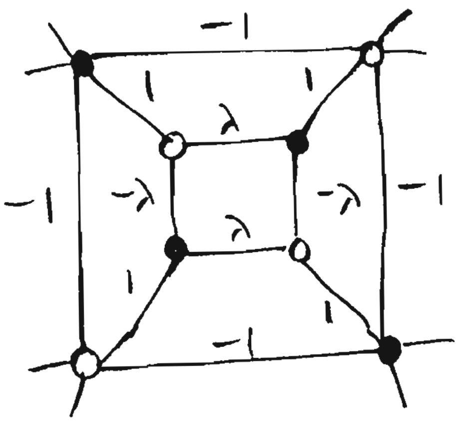



$$= \int_0^\infty \frac{(1-u^{-1})^\lambda \, du}{Q(u)}$$

$=$




By an extension of this observation we see that larger diagrams e.g.




contain within them all channels for the single box. This provides a new view of the crossing symmetry problem - closely related to the ideas for extending the simple box discussed in TN 25.

## A different construction for the contour

Having derived a meaningful formula for the result of composing two boxes, one can ask whether it could be arrived at more naturally. In fact there are probably several ways: here I will just remark on a contour construction which works the opposite way to Sparling's. Take the special case of coincident in- and out-states, i.e. AB = EF, CD = GH. Then we may first integrate out the *external* vertices and obtain:

$$\oint \frac{DUV\ \Gamma(2-\lambda-\mu)\Gamma(1+\lambda)\Gamma(1+\mu)\begin{pmatrix} A & B \\ C & D\end{pmatrix}^{\lambda+\mu}}{\begin{pmatrix} U \\ V\end{pmatrix}^{2-\lambda-\mu}\begin{pmatrix} U & A & B \\ V & C & D\end{pmatrix}^{1+\lambda+\mu}}$$

One may give explicit coordinates for $U_\alpha$ and $V^\alpha$ which diagonalise the two bilinear forms which appear, and in these coordinates it is fairly straightforward to define a contour in UV space over which the integration yields

$$\frac{\Gamma(1+\lambda)\Gamma(1+\mu)}{\Gamma(1+\lambda+\mu)}\begin{pmatrix} A & B \\ C & D\end{pmatrix}^{-2}$$

This is in agreement with the formula $\begin{pmatrix} A & B \\ C & D\end{pmatrix}^{-2}\displaystyle\int_0^\infty \frac{I_c(\lambda,\mu;u)\,du}{(1-u)^2}$

which we would obtain in this special case from the contour as constructed in the Sparling way. It seems clear (but not proved) that the different constructions do represent the same homology class. Using the second construction it is hard to generalise to AB, CD, EF, GH in general position. But it has the advantage of showing explicitly that the limits $\lambda\to 1$, $\mu\to 1$ may be regarded as corresponding to taking a contour-with-boundary.

**Other channels for the double box?**

Putting the result of the integration in the form $\displaystyle\int_0^\infty \frac{f(u)\,du}{Q(u)}$

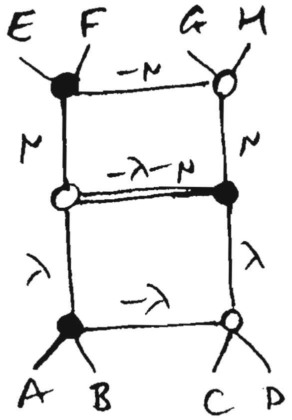
is particularly useful because it is known that this is just the amplitude which corresponds to the insertion of a ***momentum-space*** kernel of form

$$f\left(\frac{k_1\cdot k_4}{k_1\cdot k_2}\right)$$

in the appropriate channel. It follows immediately that the amplitudes in the other channels must be:

$$\int_1^\infty \frac{I_c(\lambda,\mu;u)}{Q(u)}\,du, \qquad \int_0^1 \frac{I_c(\lambda,\mu;u)}{Q(u)}\,du$$

i.e. these formulae must in some sense correspond with

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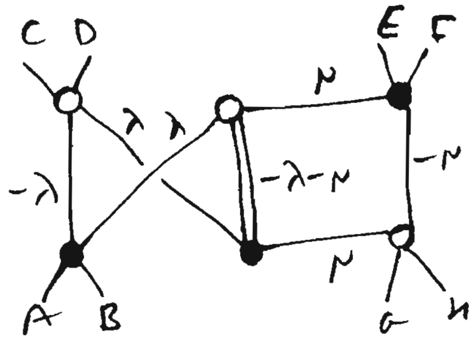
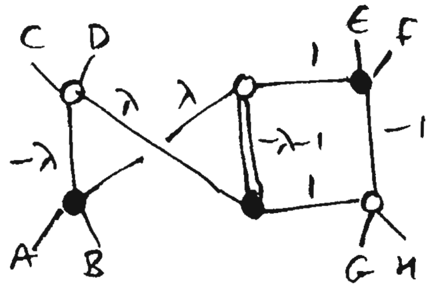



respectively. The first of these obviously cannot be a genuine contour integral as it stands, but there does seem to be a "hard contour" (in the sense explained above) which effects this correspondence. It is an open question as to whether there is a contour integral for the second of these crossed channels. If there is one, it certainly can not be obtained by integrating out the inner variables first (the required amplitude, regarded as an analytic function in $\lambda$ and $\mu$, has a pole at $\lambda+\mu = 2$; but this pole is cancelled as soon as we do the UV integral in the Sparling manner.)

Looking at the limiting case $\mu \to 1$, we note that the answers for the integrals

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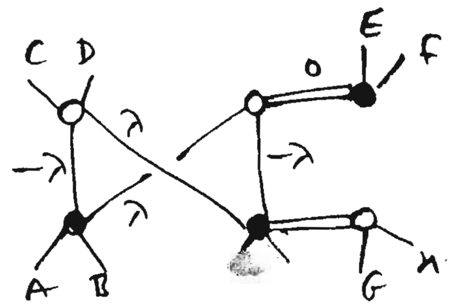
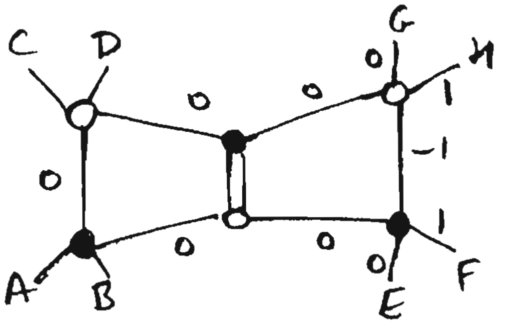



must agree, and at first sight it would appear that the contour for the double-box diagram, if it exists, must be identifiable in this limit with the contour for the twistor-transformed single-box. However, this turns out to be incorrect (see below).

**Extension to external states of non-zero helicity**

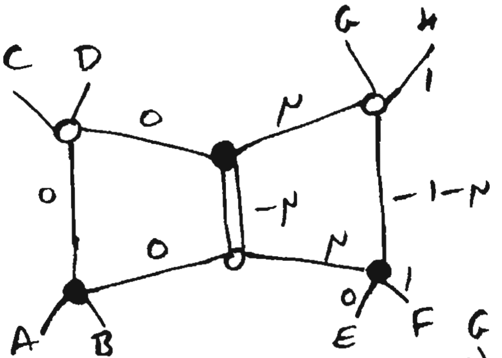
It is fairly straightforward to use spin-raising and integrating-by-parts techniques to generalise these results to non-scalar states. The first example shows that the results are not always what might be expected (by me, anyway):



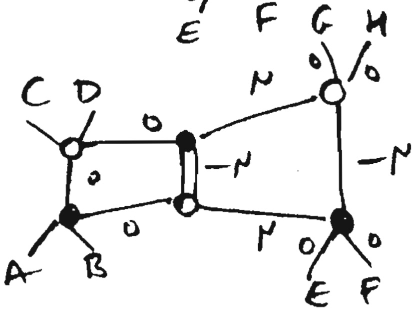

$$\equiv 0$$

Proof: this is

$$\lim_{\mu \to 0}$$



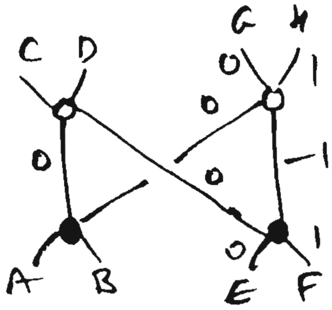
$$= \lim_{\mu \to 0} \frac{1}{\mu} \frac{\partial}{\partial F} \cdot \frac{\partial}{\partial H}$$



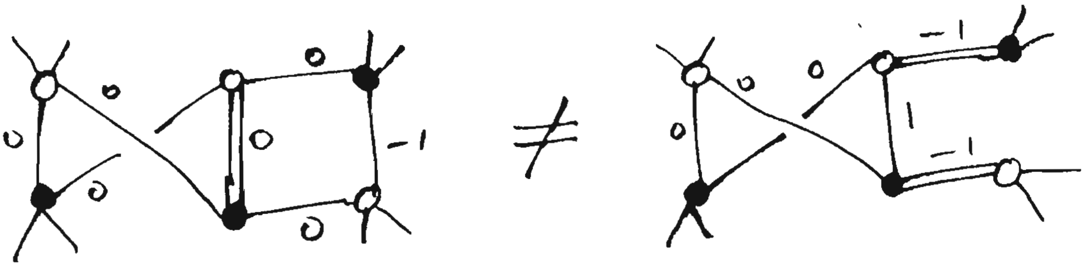
$$= \lim_{\mu \to 0} \frac{1}{\mu} \frac{\partial}{\partial F} \cdot \frac{\partial}{\partial H} \int_0^\infty \frac{du}{Q(u)}$$

$$= \lim \frac{1}{\mu} \cdot 0 \qquad = 0.$$

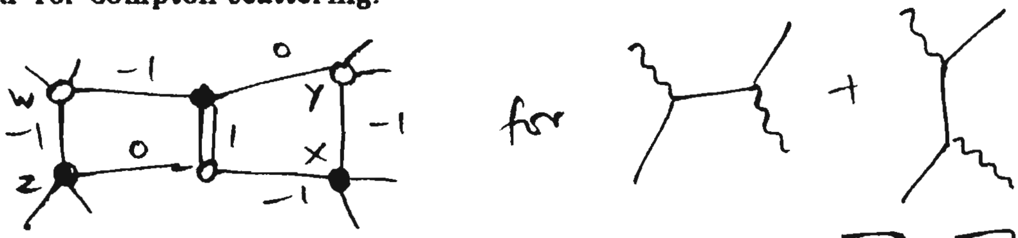
One might have expected this to be



This means that if a contour exists for the crossed channel, we shall have

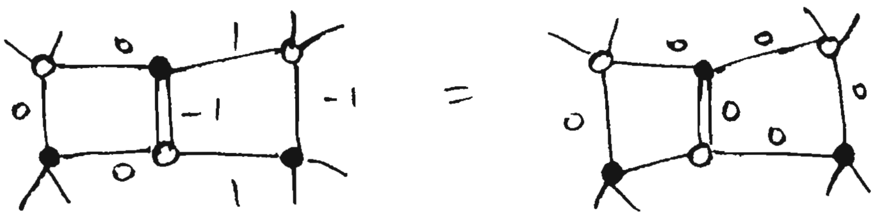


the RHS diagram being non-zero. Hence the agreement of a double box in its limiting case, and the twistor-transformed single box, will not hold in general for non-zero external helicities. Acordingly, it requires careful checking to ensure the validity of the double box originally written down by RP for Compton scattering:

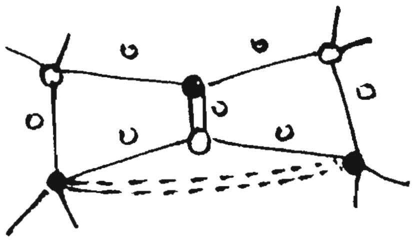


In fact this follows by operating on the double box with $\partial_W \partial_Y \; Z \partial_W$

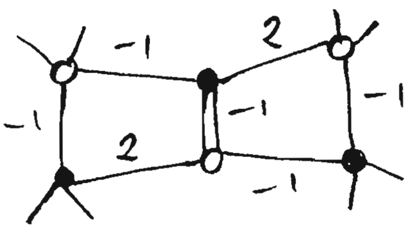
We obtain three terms, two of which cancel by virtue of the identity



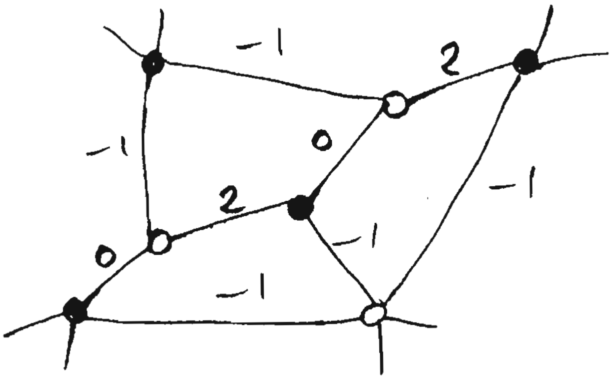
and leave



This is equivalent to agreement with the Feynman calculation. Similarly we can show that

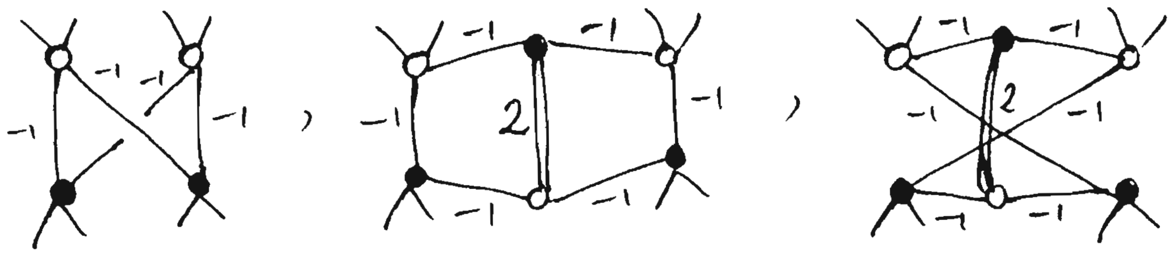


is also valid (for the appropriate channel). Hence the extended diagram




does contain within it the contours for all channels.

In a similar way we can use the double box integral to represent the "missing" channels for other first-order interactions. At this point we shall assume that the infra-red divergences may be consistently be removed by application of the inhomogeneous propagators within the double-box integral. Then of particular interest are amplitudes for SU(2) processes (see TN 23). We find for instance that the representation of pure gauge field self-interaction as a summation over




is fully justified (in this channel).

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