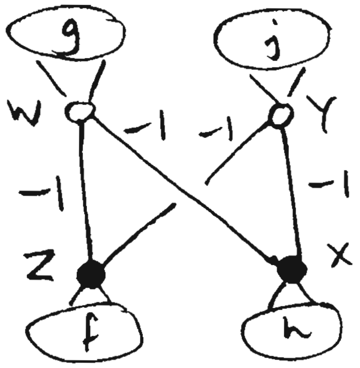
Pochhammer contours in twistor diagrams

Pochhammer contours have a role in convolving the anti-derivative twistor diagram elements in non-projective twistor space. The technicalities may be of interest in view of RP's suggestions regarding holomorphic linking. In this note I mostly consider the single box with its one channel, but the ideas naturally extend to the double box.

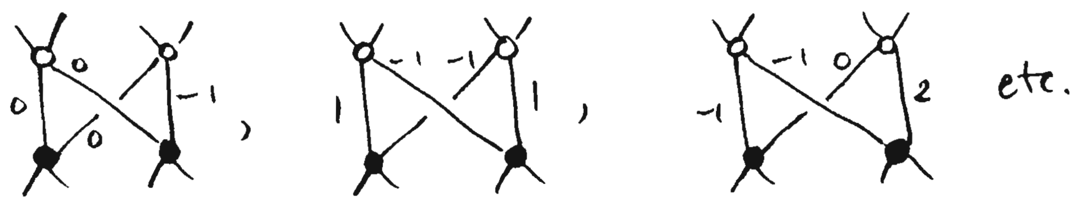
The need for Pochhammer contours is seen most directly by considering the twistor diagram:



where f, g, h, j are of homogeneity (-4)

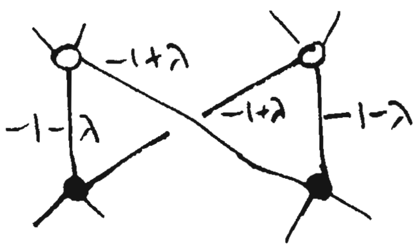
(1)

This occurs in the interaction of SU(2) gauge fields [TN 23] and so is of interest in its own right; but it is just as important to see it as the ***most general*** case of a single-box diagram with integer helicities, from which all other cases such as



can be derived.

Firstly, consider within the projective diagram calculus the limit

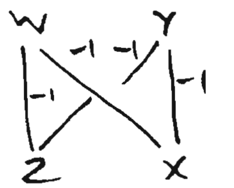
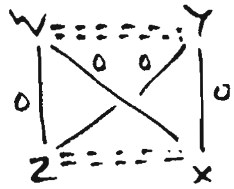
$\lim_{\lambda \to 0}$ 

(2)

It is infinite, indicating (as is to be expected) infra-red divergence. If we replace the projective diagram calculus by the inhomogeneous propagators [APH, *Proc. R. Soc. Lond.* **A397** 341-374 (1985)] we obtain a new interpretation of (1) as the integral:

$$\oint f(Z^\alpha)g(W_\alpha)h(X^\alpha)j(Y_\alpha)\{\gamma+\log(W.Z-k_1)\}\{\gamma+\log(W.X-k_2)\}$$
$$\{\gamma+\log(Y.Z-k_3)\}\{\gamma+\log(Y.X-k_4)\}\,D^4ZWXY \qquad (3)$$

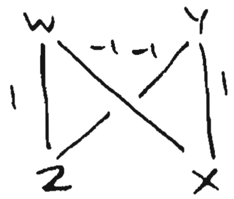
An essential point here is that we must preserve the condition

$(W\partial_Z\, Z\partial_W)(W\partial_X\, X\partial_W)$  $=$  $\qquad (4)$

Thus any proposed contour for the new integral must yield the correct "delta-function" answer when applied to the RHS diagram in (4). Similarly,

by applying $\quad (W\partial_X\, X\partial_W)^2$

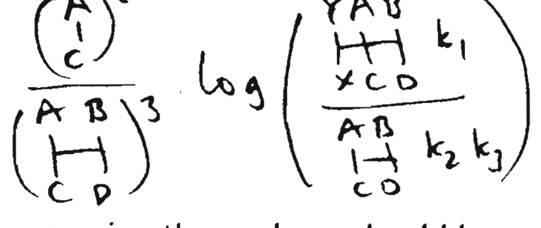
we see that the contour must give a consistent result when applied to the inhomogeneous "Møller scattering" diagram



We use this necessary condition as a guideline on how to proceed, trying to construct a contour for (3) which will meet this condition.

Let f, g, h, j be elementary, so $f(Z^\alpha) = \dfrac{2}{A.Z\,(B.Z)^3}$, $g(W_\alpha) = \dfrac{2}{W.C\,(W.D)^3}$ etc.

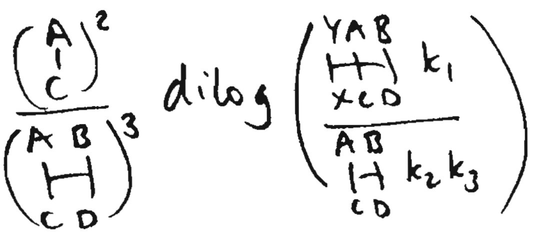
The Z^α and W_α integration can indeed be done by analogy with the "Møller" diagram, the result being essentially



which then, pursuing the analogy, should be combined with the remaining

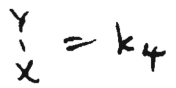
$\{\gamma+\log(Y.X-k_4)\}$ factor. In the Møller case, this factor is a double pole which could be surrounded by an S^1 . But now we have a *branch point* instead of a pole and there is in fact *no* contour corresponding to this S^1.

The Pochhammer contour saves the situation. However, to use it we have to back-track and first do the WZ integration differently, abandoning temporarily our Møller guideline. We use a contour which allows $Y_\alpha = 0$, $X^\alpha = 0$ [the computation may quite conveniently be done by expanding the WX and YZ factors in inverse powers of k]. The result of the WZ integration is then (essentially)

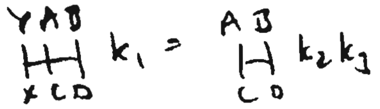


$$\left[\mathrm{dilog}\, z = -\int_0^z \log(1-t)\frac{dt}{t}\right]$$

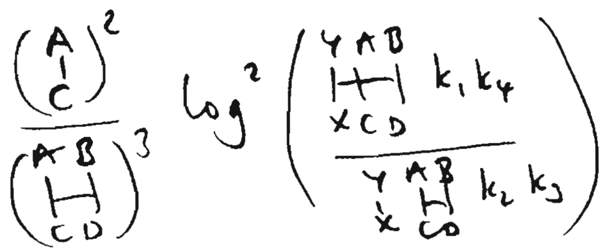
i.e. something whose *period* is the logarithmic expression we had before. Now this dilogarithm, and the remaining logarithmic factor, *can* be successfully combined by performing a Pochhammer contour integration around the branch point at



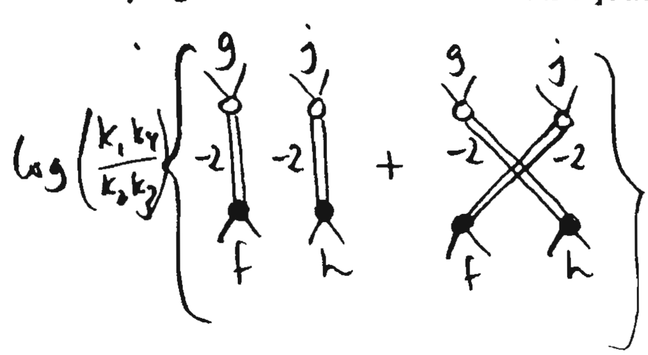
and the newly introduced branch point at



The result of this is (essentially) equivalent to the *projective* twistor factor



and the remaining integration over Y and X yields a finite answer satisfying the essential differential equation (4). It is of form:



+ {k-independent finite part}

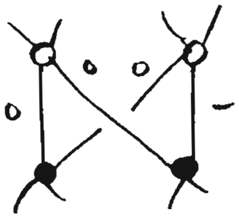
where this k-independent part is the same as we should have got from the projective twistor integral

$$\oint DWZXY \; \frac{1}{4!} \log^4\left(\frac{W.Z \; Y.X}{W.X \; Y.Z}\right) f(Z^\alpha) g(W_\alpha) h(X^\alpha) j(Y_\alpha)$$

which is itself the natural regularisation of the divergent limit (2). We must of course check that the new contour thus constructed is in fact one that could validly have been employed for the Møller diagram or for the RHS in (4); indeed this is easy to check by observing how the Pochhammer integral reduces to the residue calculus when one of the branch points happens to reduce to a pole.

Further remarks:

1. Although this construction has been motivated by requiring an extension of the diagram calculus to the (-1,-1,-1,-1) diagram, it should not be thought of as specific to this diagram. In fact, it is actually more consistent to use this construction in *all* box diagrams, since it has the effect of putting all the lines on an equal footing. This comment applies equally to diagrams like



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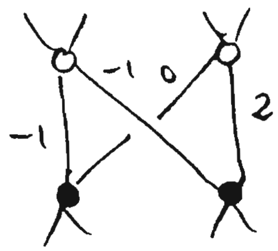
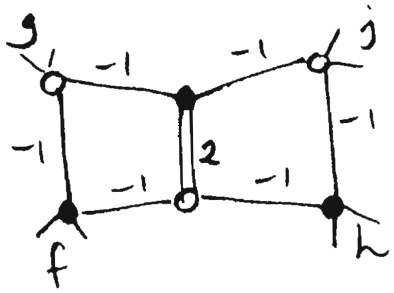


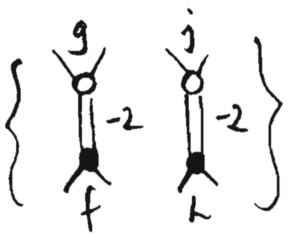
which are *not* infra-red divergent.

2. Subject to some detailed checking and computation, it should be straightforward to combine this construction with the double-box analysis and so give a finite evaluation of



and hence of the complete SU(2) interaction [TN 23].

3. There is more freedom of choice for the contour in (3) than has been indicated above. The contour described is one in which the logarithmic factors play a role only through defining branch points (equivalently, the Euler constant γ plays no role.) But we do have the freedom to add on pieces of contour which "see" the logarithm and contribute terms like

$(\gamma + \log k_2)$ 

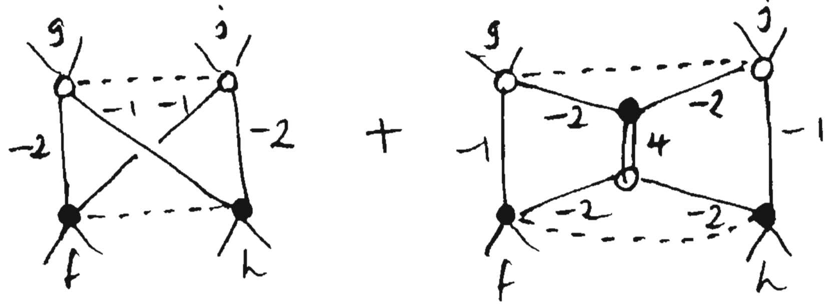
(Contours *do* play this role in the integrals which generate mass eigenstates.) However, one point of interest in the contour as originally described above is that it seems very likely to be equivalent to a contour-with-boundary construction; i.e. that the amplitude could be rewritten as

$$\oint_{W\cdot Z=k_1,\ W\cdot X=k_2,\ Y\cdot Z=k_3,\ Y\cdot X=k_4} f(Z^\alpha)\, g(W_\alpha)\, h(X^\alpha)\, j(Y_\alpha) \quad D^4 ZWYX$$

This accentuates the intriguing similarity to the *open string* calculation (TN 27)

4. The diagrams considered here are also particularly relevant to *graviton-graviton* scattering; the Feynman sum

can be written (for helicity eigenstates) as the sum of the twistor diagrams



which have the same singularity structure as the (-1,-1,-1,-1) box diagrams discussed above.

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