Twistors and the Regge Calculus

Regge calculus has been popular for many years as a method of reducing the problem of describing curved space-times to piecewise linear combinatorics. A substantial difficulty is that piecewise linear space-times satisfying Regge's field equations do not satisfy the exact Einstein vacuum equations. Indeed, the connection between the vacuum equations and solutions of Regge's equations is far from direct requiring sophisticated limiting arguments (see discussions in papers by J. Barrett etc. in C.Q.G. in the mid '80's). In this note I would like to point out that Regge space-times can indeed satisfy the Einstein vacuum equations, but only when the signature of the metric is (2,2) and the 'bones' are α- or β-planes.

The piecewise linear space-times that Regge considered have their curvature supported on codimension 2 submanifolds. The curvature is entirely due to conical singularities around these so called 'bones'. The curvature can readily be seen to have the form:

$$R_{abcd}=\epsilon\delta^2 B_{ab}B_{cd}$$

where $B_{ab}=B_{[ab]}$ is the 2-form orthogonal to the 'bone' (so that in particular B_{ab} is simple), ϵ is related to the angle deficit around the bone, and δ^2 is the δ-function supported on the bone. The Ricci curvature is thus:
$$R_{ab}=\epsilon\delta^2 B_a{}^c B_{bc}$$
For $R_{ab}=0$ we must therefore have that B_{ab} is a null 2-form: $B_{ab}=o_A o_B \varepsilon_{A'B'}$ or $o_{A'}o_{B'}\varepsilon_{AB}$ for some o_A or $o_{A'}$. Thus we see that the 'bone' is a β-plane or an α-plane respectively. This can only be real when the space-time has signature (2,2). (These ideas would be difficult to make sense of in the complex because of the use of δ-functions: one would perhaps need to develop some kind of hyper-functional interpretation.)

Example: Probably the simplest example of such a space-time is obtained by choosing a (real) β-plane through the origin aligned along o^A, and then identifying the space-time with itself dragged along the integral curves of the anti-self dual killing vector $K=x_B{}^{A'}o^{(B}o^{A)}\partial_{AA'}$.

To be more precise, choose coordinates $u^{A'}=x^{AA'}o_A$ and $x^{A'}=x^{AA'}\iota_A$ so that the metric is given by:

$$ds^2=\varepsilon_{A'B'}du^{A'}dx^{B'} \qquad \text{and} \qquad K=u^{A'}\partial/\partial x^{A'}$$

and the β-plane is given by $u^{A'}=0$. Then introduce the coordinates $\rho=u_{A'}x^{A'}$ and $\lambda=(\phi_{A'B'}u^{A'}x^{B'})/(\phi_{A'B'}u^{A'}u^{B'})$ where $\phi_{A'B'}=o_{A'}o_{B'}+\iota_{A'}\iota_{B'}$ so that $(u^{A'},\rho,\lambda)\to(u^{A'},\rho,\lambda+\epsilon)$ is the isometry generated by K. Then, we can identify λ with $\lambda+a$ for some a. (That is we can identify $x^{A'}$ with $x^{A'}+au^{A'}$ for some a.) The resulting space-time has a more interesting topology than is usually associate with 'cosmic string' type solutions (presumably $S^1\times S^1\times\mathbf{R}^2$). The holonomy around the S^1 factor caused by the identification can be checked to be $\delta_A{}^B+ao_Ao^B$

Interesting questions are:
1) What is the non-linear graviton construction for this space-time?
2) Is it possible a) to have two such bones meeting in a point, presumably there are consistency conditions. b) Is it possible to incorporate both S.D. and A.S.D Weyl curvature in a nontrivial way (i.e. with an A.S.D. bone intersecting an S.D. bone nontrivially)? It seems likely that there is no difficulty when the ASD bones miss the SD bones as will generically be the case.

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