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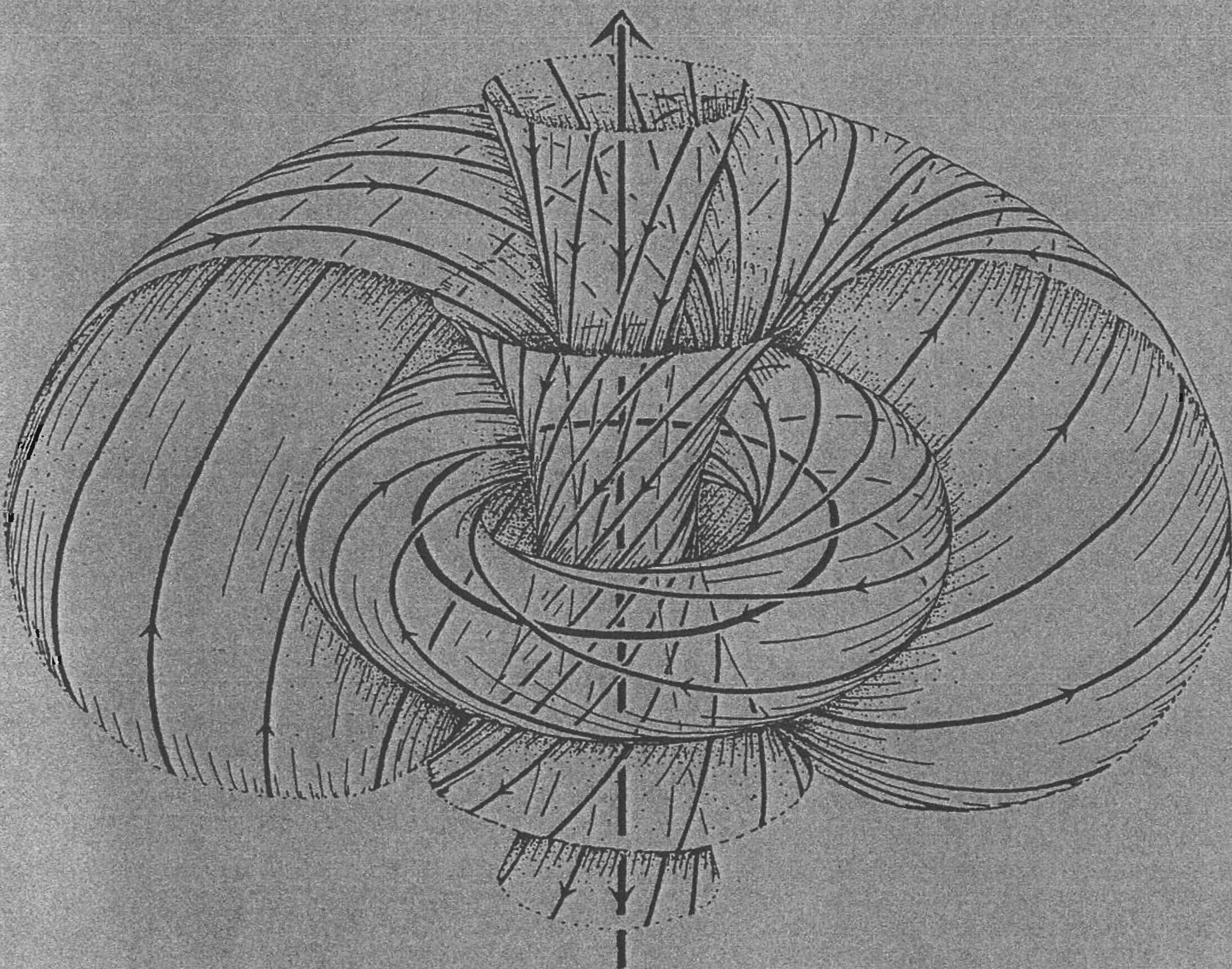

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Cohomological Residues

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This is a corrected and much improved version of an old TN article [1]. There are three sections. In §1 we generalise the dot product slightly and relate it to the connecting homomorphism α derived from the short exact sequence

$$0 \longrightarrow \mathcal{O}(-n-1) \xrightarrow{xs} \mathcal{O}(-n) \longrightarrow \mathcal{O}_{\{s=0\}}(-n) \longrightarrow 0 . \tag{1}$$

Then, in §2, we show how to describe Leray's residue map in terms of cohomology. We conclude by demonstrating a long-suspected relationship between "cohomological evaluation" and the residue map.

1. *The connecting map α and the dot product*

The sheaves in (1) are defined on the open subset X of $\mathbf{CP}^n$. s is a holomorphic function on X (such as $A_\alpha X^\alpha$) and we let S be the zero set of s in X and U be an open neighbourhood of S in X. Now we have the two maps

$$H^q(U;\mathcal{O}(-n)) \xrightarrow{r} H^q(S;\mathcal{O}(-n) \xrightarrow{\alpha} H^{q+1}(X;\mathcal{O}(-n-1)) \tag{2}$$

where r is simply restriction and α is the connecting homomorphism of (1). Let $\omega\epsilon H^q(U;\mathcal{O}(-n))$. We claim that

$$\alpha(r(\omega)) = \omega \cdot \frac{1}{s} . \tag{3}$$

Proof: We start by describing $\alpha(r(\omega))$ in terms of Čech cohomology. Let $\{\Sigma_a\}$ be an open covering of U, and $\{X_i\}$ be an open covering of X-S. Suppose

$$\omega = \{\omega_{a_0 \ldots a_q}\} .$$

To obtain $r(\omega)$ we simply regard the $\omega_{a_0 \ldots a_q}$ as having been restricted to S. The map α is in three pieces. (i) We construct a q-cochain $\tilde{\omega}$ in X (with respect to the covering $\{\Sigma\} \cup \{X\}$) as follows:

$$\tilde{\omega}_{a_0 \ldots a_q} = \omega_{a_0 \ldots a_q}$$

but $\tilde{\omega} \ldots = 0$ if the (q+1)-fold intersection includes any sets from $\{X_i\}$.
(ii) We take the coboundary of $\tilde{\omega}$:

$(\delta\tilde{\omega})_{a_0 \ldots a_{q+1}} = 0$ because ω was a cocycle,

$$(\delta\tilde{\omega})_{i a_0 \ldots a_q} = \tilde{\omega}_{a_0 \ldots a_q} = \omega_{a_0 \ldots a_q} ,$$

$(\delta\tilde{\omega}) \ldots = 0$ whenever the (q+2)-fold intersection contains more than one set from $\{X_i\}$.

(iii) We divide by s. This makes sense because $\delta\tilde{\omega}$ puts $\omega_{a_0 \ldots a_q}$ on all sets $X_i \cap \Sigma_{a_0 \ldots a_q}$ and zero on all other (q+2)-fold intersections. But this is exactly the Čech definition of

$$\omega \cdot \frac{1}{s} ,$$

where s is thought of as an element of $H^0(\{X_i\};O(-1))$.

2. *The residue map*

If φ is a holomorphic form closed on X-S and with a pole of order 1 on S then Leray's residue theorem [2] says that there exist forms ψ and θ such that

$$\varphi = \frac{ds}{s} \wedge \psi + \theta \tag{4}$$

where $\psi|_S$ is closed and holomorphic. $\psi|_S$ is called res(φ). In terms of mappings between cohomology groups this residue map comes in two parts.
(i) We think of φ as an element of $H^0(X\text{-}S;\Omega^p)$ and then we use the relative cohomology exact sequence

$$H^0(X\text{-}S;\Omega^p) \xrightarrow{c} H^1(X,X\text{-}S;\Omega^p) \rightarrow H^1(X;\Omega^p) \tag{5}$$

to map φ to a pair (ω,η) (representing $c(\varphi)$), where

$$\omega \in \Omega^{p,1}(X), \qquad \eta \in \Omega^{p,0}(X\text{-}S),$$

$$\bar\partial\omega = 0, \quad \bar\partial\eta = \omega|_{X-S} .$$

Here $(\omega,\eta) \sim (0,\varphi) \sim (\bar\partial\beta\wedge\varphi,\beta\varphi)$ where β is any C^∞ bump function identically 1 on S and with support in an arbitrary neighbourhood of S. (See [3] for the Dolbeault description of relative cohomology). (ii) We contract the normal bundle of S in X to a disc bundle

$$D \xrightarrow{\pi} S .$$

Then we squeeze the bump β until (ω,η) is supported in D. Finally we integrate ω along the fibres of π (i.e. over the discs) to get

$$\pi_*(\omega) \in \Omega^{p-1,0}(S) \ .$$

In fact this induces a map between the cohomology groups

$$\pi_*: H^1(X,X-S;\Omega^p) \to H^0(S;\Omega_S^{p-1}) \ .$$

It can be seen that if φ were of the form (4) then $\pi_*(c(\varphi)) = \psi|_S$, as required.

We can generalise the maps c and π_* (and specialise p to n = dimX) to obtain the following commutative diagram (in which the top row is exact).

$$\begin{tikzcd}
H^q(X-S;\Omega^n) \arrow[r, "c"] \arrow[dr, "\mathrm{res}"'] & H^{q+1}(X,X-S;\Omega^n) \arrow[r, "\text{forget}"] \arrow[d, "\pi_*"] & H^{q+1}(X;\Omega^n) \\
& H^q(S;\Omega_S^{n-1}) \arrow[ur, "\alpha"'] &
\end{tikzcd}$$

(Here we have also used the facts that on X we have $\Omega^n = \mathcal{O}(-n-1)$ while on S we have $\Omega_S{}^{n-1} = \mathcal{O}_S(-n)$). In particular, therefore, $\alpha_0\mathrm{res} = 0$. This result doesn't quite capture the folklore relationship between the dot product and the residue map, however. So we start again.

3. *Cohomological Evaluation and the Residue Map*

Consider

$$\omega \in H^0(X-S_1 \cup S_2 \cup \ldots \cup S_m;\Omega P) .$$

Since $X-\cup S_j$ is covered by $U_j = X - S_j$ there are various interpretations of ω by various Čech (Mayer-Vietoris) maps. The simplest case is when $m = 2$. Now the Čech map is

$$H^0(X-S_1-S_2;\Omega P) \to H^1(X-S_1 \cap S_2;\Omega P) \tag{6}$$

and the question (posed in section 2 of [4]) is: which contours in $H_p(X-S_1-S_2)$ "factor through" this interpretation? The answer (see [4]) is to consider the dual Mayer-Vietoris sequence

$$\ldots \longrightarrow H_{p+1}(X-S_1 \cap S_2) \xrightarrow{\partial_*} H_p(X-S_1-S_2) \longrightarrow \ldots \tag{7}$$

and look for contours in the image of this map.

In Dolbeault terms, the map (6) is

$$\omega \longrightarrow \omega \wedge \bar{\partial}\beta$$

where $\beta \in C^\infty(X-S_1 \cap S_2)$ and

$$\beta = \begin{cases} 0 & \text{near } S_1 \\ 1 & \text{near } S_2 \end{cases} .$$

All this is well known (and described in [4]). What was not known was its intimate relation to the taking of residues. We use the characterisation that

$$\int_{\delta\gamma} \varphi = \int_{\gamma} \mathrm{res}(\varphi)$$

(where δ is the cobord map) and the following remarkable connection between Leray's exact sequence and Mayer-Vietoris.

<u>Lemma</u>

Consider the two Leray sequences

$$\begin{tikzcd}
\cdots \arrow[r] & H_{p+1}(X-S_1 \cap S_2) \arrow[r, "\cap_a"] & H_{p-1}(S_1-S_2) \arrow[r, "\delta_a"] & H_p(X-S_1) \arrow[r] & \cdots \\
\cdots \arrow[r] & H_{p+1}(X-S_2) \arrow[r, "\cap_b"] \arrow[u] & H_{p-1}(S_1-S_2) \arrow[r, "\delta_b"] \arrow[u, equal] & H_p(X-S_1-S_2) \arrow[r] \arrow[u] & \cdots
\end{tikzcd}$$

Then the composite $\delta_b \cap_a$ is equal to the Mayer-Vietoris connecting homomorphism ∂_* in (7).

<u>Proof</u>

We use a description in terms of compactly supported differential forms, whereby classes in $H_k(M)$ are represented by closed elements of

$\Omega_c^{\dim M - k}(M)$ (see [3] for details).

In these terms,

$$\partial_*(\alpha) = \alpha\wedge d\beta$$

where β is C^∞ on $X - S_1 \cap S_2$ and

$$\beta = \begin{cases} 0 & \text{near } S_1 - S_1\cap S_2 \\ 1 & \text{near } S_2 - S_1\cap S_2 \end{cases}$$

To describe $\cap_a$, let $j\colon S_1 - S_2 \to X - S_1\cap S_2$ be the inclusion; then $\cap_a(\alpha) = j^*(\alpha)$. The description of δ_b is a little more involved.

Let D be a tubular neighbourhood of $S_1 - S_2$ relatively compact in $X - S_2$, with projection map π. Let β be C^∞ on $X - S_1\cap S_2$, chosen so that

$$\beta = \begin{cases} 0 & \text{near } S_1 - S_1\cap S_2 \\ 1 & \text{in a neighbourhood of } X - D \end{cases}$$

This is a specialisation of our earlier definition.

If $[\chi] \in H_{p-1}(S_1 - S_2)$, the form $\pi^*(\chi)\wedge d\beta$ represents $\delta_b[\chi]$.

The composite thus carries α to $\pi^* j^*(\alpha)\wedge d\beta$. Because $j_0\pi\colon D\to D$ is homotopic to the identity map, there exists an operator H such that

$$\pi^* j^* u - u = dHu + Hdu$$

for all forms u in D. Applying this to α, we find

$$\pi^* j^*(\alpha)\wedge d\beta = -\alpha\wedge d\beta = d(H\alpha\wedge d\beta)$$

and since $H\alpha\wedge d\beta$ has compact support in $D-S_1$, we see that $\pi^* j^*(\alpha)\wedge d\beta$ and $\alpha\wedge d\beta$ represent the same class in $H_p(X-S_1-S_2)$.

Comments

1. Note that S_2 could be replaced by a union of closed submanifolds $S_2,\ldots,S_m$ without any change. It is, however, essential that S_1 be a (single) closed submanifold.

2. The promised intimacy between res and ∂_* is given by the formula

$$\int_\kappa \omega = \int_{\cap_a\lambda} \mathrm{res}(\omega)$$

where $\lambda \in H_{p+1}(X-S_1\cap S_2)$ and $\kappa = \partial_*\lambda = \delta_b\cap_a\lambda$. Note that from the commutative diagram of the lemma, we have $\kappa = \partial_*\lambda$ if κ is in the image of δ_b and the image of κ in $H_p(X-S_1)$ is zero.

3. This explicit characterisation of which contours are 'cohomological' appears to be new, although widely guessed at. While suggestive, it stops short of being a complete account of the treatment of twistor diagram ears. Work is in progress.

9

References

1. SAH in TN 9.

2. Leray, J (1959) Bull. Soc. Math. France 87 81-180.

3. SAH and MAS "Relative Cohomology and Projective Twistor Diagrams" preprint.

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Abstract

Almost Hermitian Symmetric Manifolds I
Local Twistor Theory

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Abstract

Conformal and projective structures are examples of structures on a manifold which are modelled on the structure groups of Hermitian symmetric spaces. We show that each such structure has associated a distinguished vector bundle (or *local twistor bundle*) equipped with a connection (*local twistor transport*). For projective and conformal manifolds, this is Cartan's connection. The curvature of the connection provides an tensor invariant which vanishes if and only if the manifold is locally isomorphic to a Hermitian symmetric space.

Non-Hausdorff Riemann surfaces and complex dynamical systems

If twistor theory is ever to give a description of such "non-integrable" systems as the general Yang-Mills or Einstein equations, it must be able to accommodate chaotic behaviour (such as occurs in the Belinskii-Lifshitz-Khalatnikov or Misner discussion of the Bianchi type 9 vacuums). A simple type of chaotic system (~Julia sets; the Mandelbrot set) is given by iterated complex maps

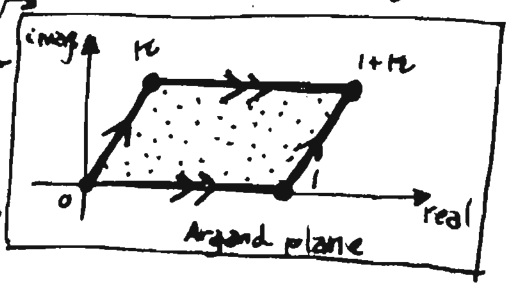
$$z \mapsto z^2 + c \qquad (z, c \in \mathbb{C},\ c \text{ const}).$$

It turns out that such maps have a close relationship with a certain type of non-Hausdorff Riemann surface.

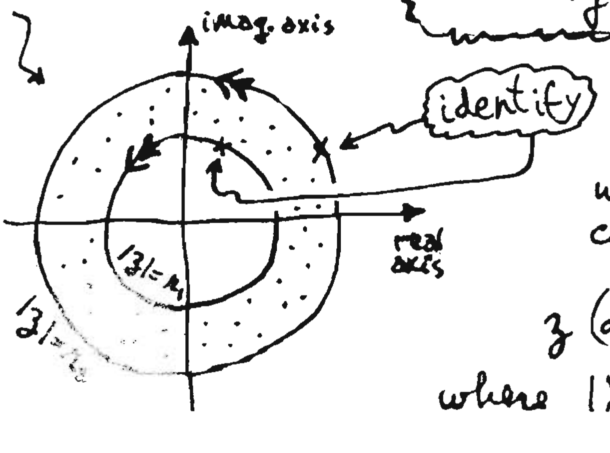
It should be noted that non-Hausdorff Riemann surfaces have already found significant application in the Woodhouse-Mason description of Ward's construction for stationary axi-symmetric solutions of Einstein's vacuum equations, so it is quite possible that non-Hausdorff surfaces of the type that I am considering may also find some significant role within twistor theory.

Let us first recall how an ordinary Riemann surface of genus 1 (torus) may be constructed:

identify along ↗ and along ↠



An alternative procedure would be:



where the strip $r_1 \leq |z| \leq r_2$ is curled into a torus by identifying

z (at $|z| = r_1$) with λz (at $|z| = r_2$),

where $|\lambda| = r_2 / r_1$. Thus, in a certain sense, this Riemann surface codes the information of the map

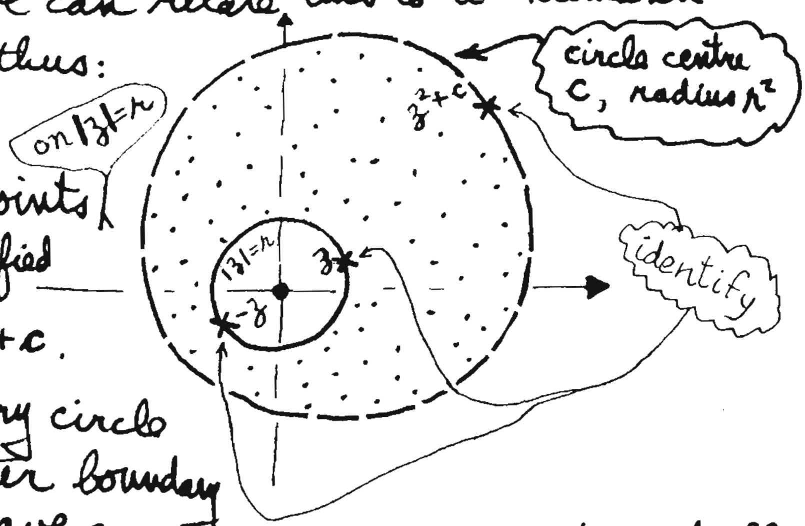
$$z \mapsto \lambda z,$$

where we go once around , and iterations of this map correspond to going many times around.

Now consider the quadratic map

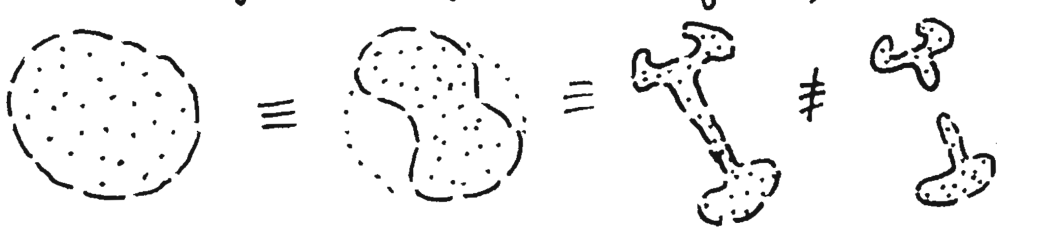
$$z \mapsto z^2 + c.$$

Similarly to the above, we can relate this to a "Riemann surface" S, constructed thus:



Here the pair of opposite points z and $-z$ are to be identified with the single point z^2+c. Taking the inner boundary circle to be present and the outer boundary circle to be absent, we have constructed a non-Hausdorff Riemann surface. (Each point has a neighbourhood that is a small complex open disc, but on $|z|=r$, the discs for z and for $-z$ always overlap.)

However, this singles out $|z|=r$ as "special", which we do not want, so we must adopt a viewpoint that removes this feature. It has already been apparent that some new concept of "complex manifold" is needed in twistor theory, in which we have an equivalence (for non-compact complex manifolds)



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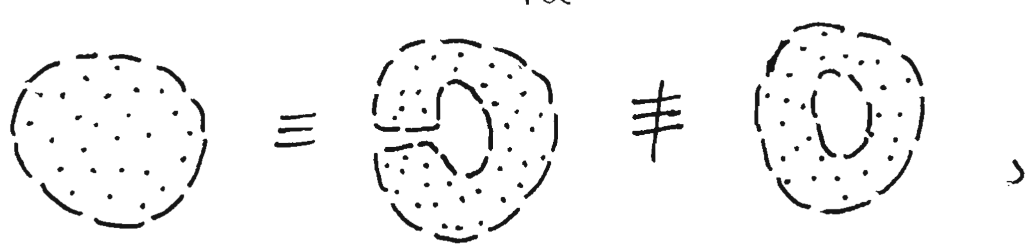
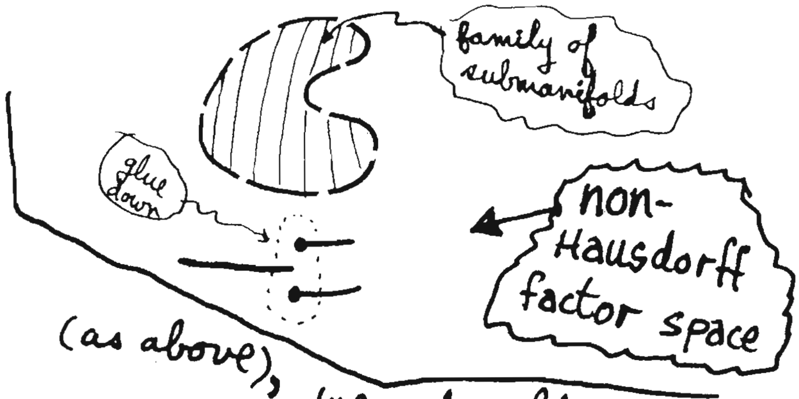


, etc.

Thus, the space can be "pared" away or built up at its (non-compact) edge, but not so as to give rise to a change in its global structure. The concept remains somewhat vague, as of now, but something of this sort is needed if we require elements of

$$H^1(X, \Theta) \quad \text{holomorphic vector fields}$$

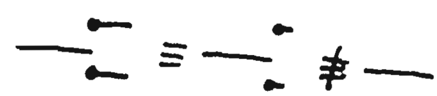
to represent the infinitesimal deformations of a non-compact complex manifold X. (Deformations of the "boundary" don't show up in $H^1(X,\Theta)$.) Now consider how a non-Hausdorff complex manifold can often arise (as indeed is the case in the Woodhouse-Mason construction):



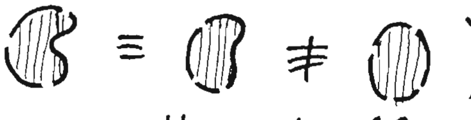
If we have:

(as above), we should also have:

; and also

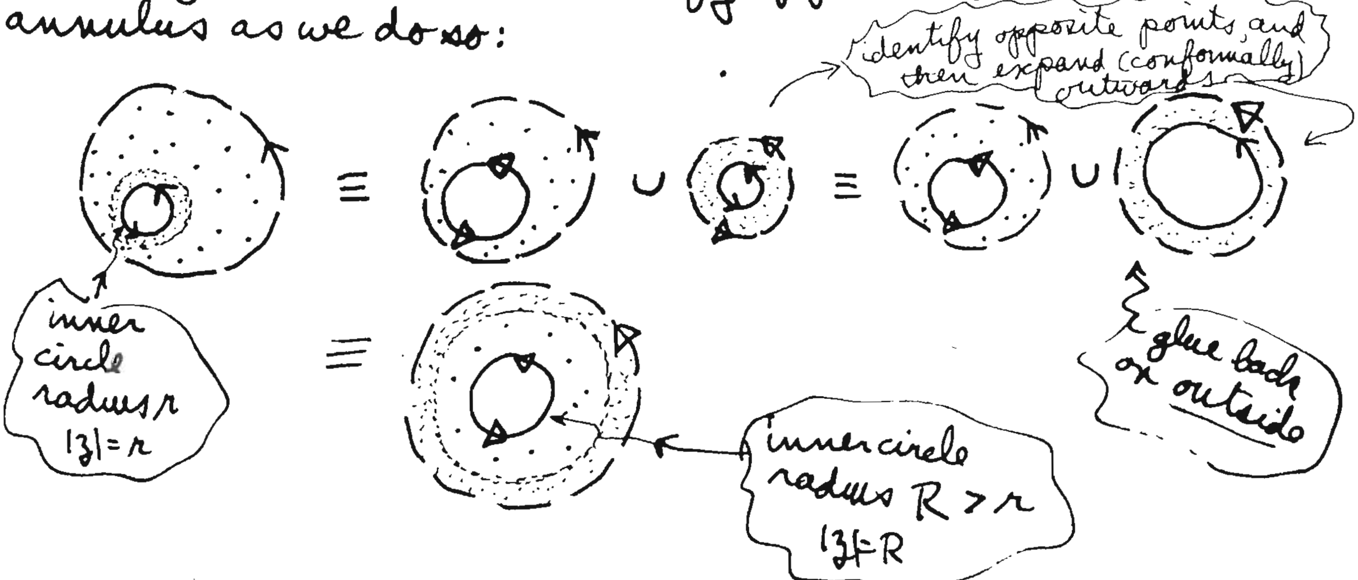


(since



). Thus we can "split" or "reglue" non-Hausdorffness, provided that (in some appropriate sense) we do not alter the "global connectivity" of the space. (I think that this can be stated in terms of what local deformations are allowed and what are not — but this needs more understanding.)

In the case of the non-Hausdorff Riemann surface just constructed, we can remove a small annulus from the inner circle boundary and glue it on at the outer circle boundary, but we must identify opposite points of this annulus as we do so:



Also, we can perform this operation in reverse so long as there remains room on the inside.

If c is in the Mandelbrot set M, then the topological structure of S (under this equivalence) appears to differ in an essential way from when c is not in M. For in some sense S has a "covering space" which is the complement K of the Julia set in $\mathbb{C}$, and whether c is in M or not depends on the multiple-connectivity properties of K. (When $c \notin M$, K has a Cantor set removed from it, but when $c \in M$, K is topologically an annulus.)

Some clarifying ideas seem to be needed.

Roger Penrose

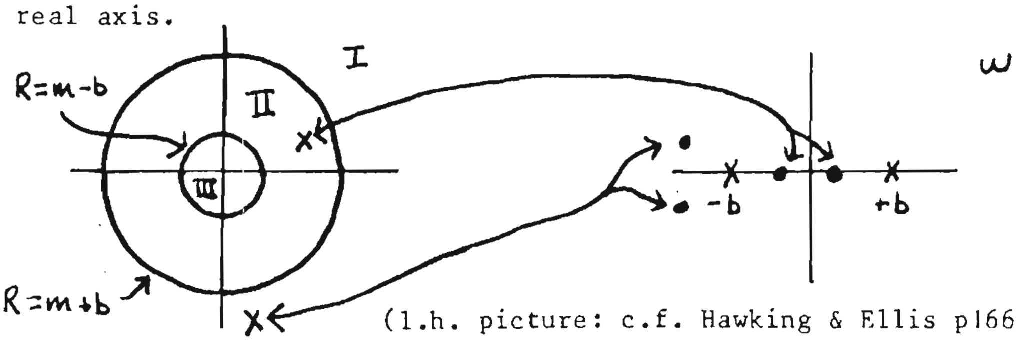
MORE ON THE TWISTOR DESCRIPTION OF THE KERR SOLUTION

In my article in TN 27, I outlined the relationship between the non-Hausdorff twistor spaces arising from NMJW's and LJM's construction and the geometry of the Kerr and Schwarzschild solutions. The purpose of this note is to expand one or two points that arose there.

Recall that the space of orbits of the two Killing vectors in the Kerr solution corresponds to the space of quadratic maps $p: X \to R_U$, where X is a copy of $\mathbb{CP}^1$ (with coordinate q) and R_U is the reduced twistor space consisting of two Riemann spheres (coordinate w) which are identified everywhere except for the pairs of points at infinity and at $w = \pm b$. In order to determine a map p, we need to know first the values of w for which the discriminant of the equation $w = p(q)$ vanishes, and then which point of X is mapped to each of the pair of points at both $w = +b$ and at $w = -b$. If we write p in the form

$$p(q) = \tfrac{1}{2}r(q^{-1} - q) + z,$$

then the two values of w are $z + ir$ and $z - ir$, where z and r are the usual Weyl coordinates; and p is determined by these and the choice of one of four possible treatments of the double points. Orbits of the Killing vectors outside the outer horizon or inside the inner one are given by real z and real, positive r, and therefore correspond to pairs of complex conjugates in the w-plane. Orbits between the two, on the other hand, have z again real but r purely imaginary; moreover z and r are constrained so that the points $z \pm ir$ lie between +b and -b on the real axis.

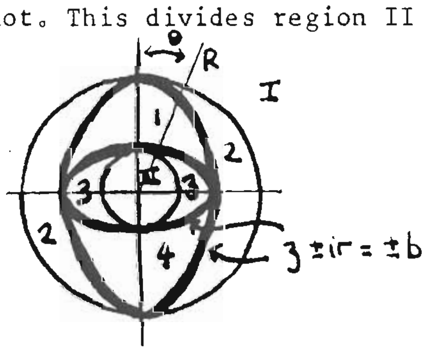


(l.h. picture: c.f. Hawking & Ellis p166)

There are, however, some values of z and r for which we cannot evaluate the metric directly by following the Ward splitting procedure. As NMJW and LJM showed in their paper, the method works provided the points $w = z + ir$ and $w = z - ir$ are distinct, and are both places where the two w-spheres are identified.

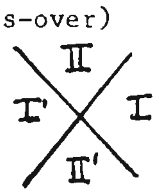
In the ordinary outside region I, therefore, the only problems can occur as $r \to 0$; and in their paper NMJW and LJM found the conditions on the bundle over R_U for the metric to be well-behaved on the axis of the Weyl coordinates (which corresponds to either an axis or a horizon in the space-time). Similarly, in region III, where the manifold can be continued analyticly out to $\mathscr{I}$, there are the two parts of the axis and the horizon, and, in addition, the ring singularity. In my previous article, I mentioned the conjecture that this might correspond to a map p for which the pull-back of the bundle over R_U to one over X is non-trivial. This does in fact turn out to be the case, and it can be shown that when $z = 0$ and $r = a$, the pulled-back bundle is $L_2 \oplus L_{-2}$.

By contrast, in region II we can have values of z and r such that one of the pair $z \pm ir$ coincides with one of $\pm b$, but the other one does not. This divides region II into four, as follows:



(volumes 2 and 3 are of course connected since there is rotational symmetry about the z-axis.)

and these four volumes correspond precisely to the four different maps p that exist for each pair (z,r), and thus to the four different possible treatments of the double points $\pm b$. This means that each pair of points on the real axis in the w-plane between +b and -b represents four orbits of the two Killing vectors in the space-time; and if we consider the analytic continuation of the space-time (putting in the point at the $R = m + b$ cross-over)



where regions I' and II' are isometric to I and II in the usual way, we actually have eight orbits for each pair. (In I and II, we define r by $r = -\frac{1}{2}i(w_1 - w_2)$; in I' and II', we take $r = +\frac{1}{2}i(w_1 - w_2)$.)

This raises various questions. If each pair of points (w_1, w_2) on the real axis between +b and -b corresponds to four orbits in the space-time, why is the same not also true of each pair of complex conjugates, or even for each general pair of points, in the w-plane? Secondly, how can we tell that the space-time is in fact regular across the hypersurfaces where one of the points coincides with +b or -b; and what do these surfaces mean geometrically?

The answer to the first question is that in the complexification of the Kerr solution each (ordered) pair of points does represent four Killing vector orbits, but not all of these intersect the real slice which is the space-time. Thus there are four real orbits for pairs in (-b,+b) on the real axis, two real orbits (one in region I and one in region III) for pairs of distinct complex conjugates, and none otherwise. Trying to find another real orbit for the pairs of conjugates would be equivalent to interpreting the axis in volume 1 of region II as a horizon; and this would be incompatible with regularity at the orbit $w_1 = +b = w_2$. Outside the outer horizon we are forced to think of $r = 0$ as an axis since it is the space-like Killing vector which vanishes on it. If $r = 0$ were a horizon then J, the metric on the space of orbits, would change signature to (+,+) across it.

We can also use the analyticity of the complexification to see that J is well-behaved across the boundaries between volumes 1,2,3 and 4. By considering small variations of z into the complex, we can move (z,r) from one volume to another without $z \pm ir$ coinciding with $\pm b$, but with the explicit effect of changing the treatment of the double points by the corresponding maps $p:X \rightarrow R_U$. This is made clear by the behaviour of the open sets covering X on whose overlaps the pull-back of the bundle over R_U is described by the pull-backs of the patching matrices in the standard form I described before.

Finally, the boundaries themselves in fact represent the light-cones of the two points where the axis and the (outer) cross-over intersect.

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NMJW and LJM Nonlinearity <u>1</u> 73-114 (1988)

JF 'Non-Hausdorff Twistor Spaces for Kerr and Schwarzschild' TN <u>27</u>

SW Hawking & GFR Ellis 'The Large Scale Structure of Space-time' CUP 1973

Thanks to NMJW.

James Fletcher.

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On Ward's Integral Formula for the Wave Equation in Plane Wave Space-Times

L. J. Mason

In [1], Ward presents an integral formula for the general solution of the wave equation in plane wave space-times. The purpose of this note is to show how this relates to the twistor integral formula in flat space, and to generalize the formula to arbitrary helicity. The generalization of the formula shows how it is that Maxwell theory satisfies a kind of Huygens principle in plane wave space-times. This suggests further generalizations of Huygens principle. However, electromagnetic fields in plane wave space-times provide the only nontrivial example of such generalized Huygen's principles.

Ward's integral formula:
Consider the plane wave space-time with metric:

$$ds^2 = du\cdot dv - G_{ij}(v)dx^i\cdot dx^j \qquad i,j=1,2.$$

For convenience we shall choose a conformal scale such that $det(G_{ij})=1$ (all our considerations will be conformally invariant, so this doesnt involve any loss of generality). The hypersurfaces of constant u are null and support δ-function solutions, $\delta(u)$, of the wave equation, (this follows from $\Box u=0=(\nabla_a u)(\nabla^a u)$.

The hypersurfaces of constant $u_b=u+2x^i b_i+F^{ij}b_ib_j$ (where $F^{ij}=\int G^{ij}(v)dv$) are related to the hypersurfaces of constant u by the symmetries

$$(u,\ v,\ x^i)\to(u+2x^i b_i+F^{ij}b_ib_j,\ v,\ x^i+F^{ij}b_j),$$

and so also support δ-function solutions of the wave equation. We can form the general solution of the wave equation by averaging over these δ-function solutions:

$$\phi(x)=\int\Phi(u_b,b_i)\mathrm{d}^2b$$

where Φ is an arbitrary function of its three arguments. When $G_{ij}(v)=\delta_{ij}$, (flat space) the formula reduces to the Whittaker integral formula.

Relationship with the twistor integral formula
In flat space, this formula can be seen to be the twistor integral formula as follows. Write:

$$\pi_{1'}/\pi_{0'}=\zeta=b_1+ib_2, \qquad \bar\pi_1/\bar\pi_0=\bar\zeta=b_1-ib_2.$$

Then $u_b=x^{AA'}\pi_{A'}\bar\pi_A/\pi_{0'}\bar\pi_0=\omega^A\bar\pi_A/\pi_{0'}\bar\pi_0$ and we can put:

$$\Phi\mathrm{d}^2b\equiv(\bar\pi_0\pi_{0'})^{-2}\Phi(\omega^A\bar\pi_A,\pi_{A'},\bar\pi_A)\bar\pi^A\mathrm{d}\bar\pi_A\wedge\pi^{A'}\mathrm{d}\pi_{A'}$$

so that $(\bar\pi_0\pi_{0'})^{-2}\Phi(\omega^A\bar\pi_A,\pi_{A'},\bar\pi_A)\bar\pi^A\mathrm{d}\bar\pi_A$ is a homogeneity degree -2 Dolbeault representative on twistor space constructed from the characteristic data at $\mathscr{I}$, Φ, for the field ϕ as in my TN article [2]. A.

A difficulty with the twistorial interpretation of this formula in the curved case is that the appropriate complex structure on the space of primed spinors (on which b_i are coordinates) shifts as v varies; $\pi_{1'}(v)/\pi_{0'}=\zeta(v)=\bar m^i(v)b_i$, the b_i are held constant. The complex structure is determined by the 2-metric $G_{ij}(v)=\bar m_{(i}m_{j)}$. It is therefore not clear how one can obtain a global holomorphic interpretation of the formula in the conformally curved case. (One can, of course, provide a holomorphic interpretation of the formula on each of the hypersurface twistor spaces based on hypersurfaces of constant v; the above formula then answers, to a certain extent the question of how to identify cohomology classes based on one hypersurface with those on subsequent hypersurfaces.)

Generalization to higher helicity

Plane waves, and in fact all Brinkman waves, have a covariantly constant spinor, o^A, aligned along the generators of the hypersurfaces of constant v. This can be used to raise and lower helicity of massless fields.

If $\phi(x)$ is a solution of the wave equation, then $\varphi_{AB}=o^{A'}\nabla_{AA'}o^{B'}\nabla_{BB'}\phi(x)$ is an ASD solution of the Maxwell equations. All solutions of the Maxwell equations can be put in this form (this follows from $[o^{A'}\nabla_{AA'}, o^{B'}\nabla_{BB'}]=0$ together with $o^{A'}\nabla^A_{A'}\varphi_{AB}=0$ from Maxwell's equations). Similarly, all solutions of the neutrino equations can be put in the form $o^{A'}\nabla_{AA'}\phi(x)$. Higher helicity fields constructed in this way will not, in general, satisfy the Z.R.M. equations because of Buchdahl conditions. However, there is a consistent potentials modulo gauge description

$$\psi_{AA'_1\cdots A'_{n-1}}=o_{A'_1}\cdots o_{A'_{n-1}}o^{B'}\nabla_{AB'}\phi(x)$$

satisfies the $(n-1)$-potential equations.

This description leads to the following formula for the general solution of Maxwell's equations:

$$\varphi_{AB}(x)=\int o^{A'}\nabla_{AA'}\, o^{B'}\nabla_{BB'}\Phi(u_b,b_i)d^2b.$$

Note that the first ∇ in this expression acts on the free spinor index on the second ∇ as a *covariant* derivative. Let $\partial_{AA'}$ denote the coordinate derivative in the spinframe determined by the null tetrad $l=dv$, $n=du$, $m=m_i dx^i$, $\bar{m}=\bar{m}_i dx^i$ where $m_{(i}(v)\bar{m}_{j)}(v)=G_{ij}(v)$ and the phase of m_i is determined by the condition that $\dot{m}_{[i}\bar{m}_{j]}+\dot{\bar{m}}_{[i}m_{j]}=0$ (the dot, $\cdot$, denotes $\partial/\partial v$). (Note that this last condition together with $\det(G_{ij})=1$ implies that $\dot{m}_i=\bar{\sigma}\bar{m}_i$ for some $\bar{\sigma}(v)$.) Then the spin coefficients are just $\gamma_{aB}{}^{C}=\sigma\iota_{A'}o_A o_B o^C$ and $\psi_4=\dot{\sigma}$. This formula then becomes, using coordinate derivatives in the above spin frame:

$$\varphi_{AB}(x)=\int\left\{o^{A'}\partial_{AA'}\, o^{B'}\partial_{BB'}\Phi(u_b,b_i)+\sigma o_A o_B(\partial_{u_b}\Phi)\right\}d^2b.$$

For higher helicity, the 'field' versions of these formulae fail to make reasonable sense because of Buchdahl conditions, however the potentials modulo gauge formulae do make sense.

[In the flat case, $\sigma=0$, write $\bar{\pi}_A=\bar{\pi}_0 o^{A'}\partial_{AA'}u_b$ and $\bar{\alpha}=(\bar{\pi}_0)^{-4}(\partial^2_{u_b}\Phi)\left(\overline{(\bar{\pi}_0)^{-2}\bar{\pi}^A d\bar{\pi}_A}\right)$. Then the above becomes the Dolbeault version of the (-4)-homogeneity complex conjugate (dual) twistor integral formula

$$\varphi_{AB}(x)=\int\bar{\pi}_A\bar{\pi}_B\bar{\alpha}\wedge\bar{\pi}_A d\bar{\pi}^A.$$

If we put $\frac{\partial}{\partial\omega^A}=(\pi_{0'})^{-1}o^{A'}\partial_{AA'}$, the formula generalizes the 0-homogeneity twistor integral formula.]

Huygen's principle

In Penrose (1972) it is demonstrated that a δ-function solution of the Maxwell's equation in a conformally curved plane wave space-time must also have a 'tail'. One can produce such solutions easily using the above ideas. Pick a null hypersurface, $u=0$. Let $\theta(u)$ be the Heavyside function, $\theta(u)=1$ for $u>0$, and $\theta(u)=0$ for $u<0$. Since any function of u is a solution to the Laplacian, we have in particular that $\phi=u\theta(u)$ is a solution. This means that $A_{AA'}=o_{A'}o^{B'}\nabla_{AB'}\phi=o_{A'}\iota_A\theta(u)$ is a vector potential solution to Maxwell's equations. The corresponding field is

$$\varphi_{AB}=\iota_A\iota_B\delta(u)+o_Ao_B\sigma\theta(u)$$

which has the tail $o_Ao_B\sigma\theta(u)$.

If, instead, we start with $\phi=\theta(u)$ we obtain a δ-function vector potential, $A_{AA'}=o_{A'}\iota_A\delta(u)$, without a tail, and so the field, $\varphi_{AB}=\iota_A\iota_B\delta'(u)+o_Ao_B\sigma\delta(u)$, also has no tail. This would seem to suggest a generalized form of Huygen's principle in which it is sufficient to have solutions supported on light cones or (as we have shown above) solutions supported on a family of null hypersurfaces such that there is at least one hypersurface in the family normal to each null direction through each point. The relevant solutions may be a sum of the first n derivatives of δ-functions thus leading to a hierarchy of Huygen's principles, $\mathcal{H}^n$, in which for the n^{th}, only the first n derivatives of δ-functions are allowed.

An alternative formulation of Huygen's principle is that if one poses initial data on some hypersurface Σ, then the solution at a point p depends only on the data at the intersection of the light cone L_p of p with Σ. The generalizations would then seem to correspond to requiring that the solution at p depends only on the data at $L_p\cap\Sigma$ and its first n derivatives. It would be interesting to find an example of $\mathcal{H}^\infty$.

Many thanks to George Sparling for discussions.

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LJM. TN. # 22

Hypersurface Twistors

L J Mason

The purpose of this note is to describe some results concerning hypersurface twistors and initial data. The first results describe the extent to which hypersurface twistor spaces can be used to encode initial data. In particular I discuss how the situation changes significantly when the hypersurface is chosen to be a light cone or at infinity, where, roughly speaking, half the data is lost. This fact leads to some of the difficulties one has with the 'Googly'. I also discuss (the failure of) a twistorial definition of positive frequency for initial data sets on hypersurfaces.

Encoding initial data
Before one can start using and applying hypersurface twistors, there arises the question of how they can be used to encode initial data sets for the Einstein vacuum equations. This question can be divided into two parts. The first part is how the conformally invariant part of the initial data set can be encoded (this consists of the conformal structure on a three manifold and the trace free part of the extrinsic curvature). The second part is concerned with encoding the conformal factor of the three metric and its time derivative, the trace of the extrinsic curvature.

There are (at least) two approaches to encoding initial data into hypersurface twistor spaces. I refer to one approach as the 'real' approach, and the other as the 'complex' approach. Let Σ be the complexification of a real space-like hypersurface $\Sigma_{\mathbb{R}}$ in an analytic Lorentzian space-time M. Let $P\mathcal{T}$ be the hypersurface twistor space, and PN be the (real) codimension one hypersurface in $P\mathcal{T}$ whose points correspond to hypersurface twistors which intersect $\Sigma_{\mathbb{R}}$. In the real approach one is allowed to know the location of the hypersurface PN in $P\mathcal{T}$ whereas in the complex approach one is only allowed $P\mathcal{T}$ as a complex manifold and holomorphic structures thereon.

The real approach: This is now relatively well understood, Sparling (1983) LeBrun (TN9,1984&1985), Penrose (1984), Mason (1985). The real approach is relatively easy to compute with as calculations can all be performed locally on PN using the Chern-Moser connection. LeBrun showed that PN as a CR manifold determines $\Sigma_{\mathbb{R}}$ and the conformally invariant part of the initial data. In Mason (1985) it was also shown that one could encode the information of the conformal factor if one introduced a homogeneity degree two (1,0)-form, ι, which generalizes $I_{\alpha\beta}Z^\alpha dZ^\beta$ from flat space twistor theory. The constraint equations could then be articulated as $I_{[\alpha\beta;\gamma]}=0$. In order to obtain a formula for the evolution, it was necessary to introduce a further homogeneity degree two (1,0)-form, σ, which generalizes $H_{\alpha\beta}Z^\alpha dZ^\beta$; σ encodes the information of the location of the hypersurface Σ in M^4, in flat space a point $X^{\alpha\beta}$corresponds to a point of Σ iff $X^{\alpha\beta}H_{\alpha\beta}=0$. This approach has various defects; from a twistorial point of view, the data depends on free functions of 5 variables, as compared to 3-variables for the gravitational field initial data. The characterization of those CR manifolds corresponding to gravitational initial data sets requires the knowledge of the location of the $\mathbb{CP}^1$'s in PN correponding to points of $\Sigma_{\mathbb{R}}$. These are then determined locally using the Chern Moser connection. These facts substantially limit the applicability of this approach.

The complex approach: This has not been much studied, but has presumably been around in folklore for some time. The basic observation is that, using a generalization of the nonlinear graviton construction, $P\mathcal{T}$ can be seen to be the twistor space of a 4-dimensional conformal manifold, $\mathcal{M}^4$, with ASD Weyl curvature into which Σ^3 is embedded together with its conformal equivalence class of initial data (conformal 3-metric and trace free part of the extrinsic curvature), LeBrun (1979). The space $\mathcal{M}^4$ is colloquially known as 'heaven on earth'. In order to capture the conformal initial data set, one must encode the information of the location of Σ^3 in $\mathcal{M}^4$.

When the extrinsic curvature is pure trace, the location of Σ can be encoded by means of a global holomorphic homogeneity degree 2 1-form, σ, on $P\mathcal{T}$ (σ vanishes on restriction to those holomorphic curves corresponding to points of Σ). When the extrinsic curvature is general it appears to be impossible to encode the location of Σ in $\mathcal{M}$ using local holomorphic structures on $P\mathcal{T}$; the σ as defined in the 'real approach' is no longer holomorphic. However, it is straightforward to encode the location of Σ using a cohomology class which, abusing notation, we can also call σ. The cohomology class σ can be taken, for instance, to be the element of $H^1(P\mathcal{T},\mathcal{O}(-2))$ corresponding to the solution of the wave equation on $\mathcal{M}^4$ which is zero on Σ and whose normal derivative is some lapse function. As a consequence we have that for space-like hypersurfaces the conformal equivalence class of initial data is encoded in $P\mathcal{T}$ together with σ. The conformal factor can be similarly encoded by means of the cohomology class $\iota \in H^1(P\mathcal{T},\mathcal{O}(-2))$ which corresponds to the solution of the wave equation which is 1 in the desired conformal scale.

The holomorphic approach has the important advantage that the twistor data consists of effectively free functions of three variables. I have not as yet been able to articulate the constraint and evolution equations in this context. Insight into this problem would perhaps be obtained from relating the real and complex approaches; the real and complex approaches should be related in much the same way as Dolbeault is related to Čech cohomology. However, one may need to use more sophisticated cohomology classes for ι and σ in the holomorphic approach such as elements of $H^1(P\mathcal{T},\Omega^1(2))$.

Light cones and $\mathscr{I}$: The canonical hypersurface twistor spaces are those where the hypersurface is taken to be one of past or future null infinity. If the above results were to hold for $\mathscr{I}$, then the structures σ and ι would coincide, thus reducing the complexity of the description. Another way to reduce the complexity of the description is to use light cones as initial data hypersurfaces since then the information of the location of the hypersurface is encoded simply as the quadric, $Q \subset P\mathcal{T}$, whose points correspond to the generators of the null cone, $\mathcal{N}$. (Often, when defining hypersurface twistor spaces for null hypersurfaces, Q is deleted from $P\mathcal{T}$. It can be checked that Q embeds holomorphically in $P\mathcal{T}$. I am including Q since deleting it only reduces the amount of information available.) There are three cases to consider; null $\mathscr{I}$, space-like $\mathscr{I}$ and a light cone $\mathcal{N}$. Unfortunately, in all these cases half the initial data is lost. (This is of particular irritation when one hopes to use asymptotic twistor space as the basic twistor space for the googly construction.)

Space-like $\mathscr{I}$: The hypersurface twistor space construction encodes, as usual, the intrinsic conformal structure of $\mathscr{I}$ and the (trace free part of) the extrinsic curvature which vanishes. However the free asymptotic data consists of the intrinsic conformal structure of $\mathscr{I}$ and its *third* derivative into the space-time (the first derivative of the electric part of the Weyl curvature at $\mathscr{I}$). We therefore see that 'half' the data, the third derivative of the conformal structure, is lost.

'***Heaven in church***': This is the colloquial name for the heaven construction based on a light cone. As in the other heaven constructions, one obtains a space-time $\mathcal{M}^4$ with ASD Weyl curvature into which the hypersurface $\mathcal{N}$ is embedded. The 'heaven', $\mathcal{M}^4$, is constructed as the space of holomorphic curves with S^2 topology in $P\mathcal{T}$ and $\mathcal{N}$ consists of those curves which intersect the quadric Q in $P\mathcal{T}$. The hypersurface $\mathcal{N}$ acquires initial data from its embedding in $\mathcal{M}^4$. However, this initial data is not the original set. In order to see this, consider the case where the quadric, Q, in $P\mathcal{T}$ can be blown down to a line L in some complex manifold $\widetilde{P\mathcal{T}}$ with $P\mathcal{T}\setminus Q\simeq\widetilde{P\mathcal{T}}\setminus L$. This then implies that the 'heaven in church' construction embeds the null hypersurface, $\mathcal{N}$, as a light cone in $\mathcal{M}^4$ as $\widetilde{P\mathcal{T}}$ can be taken to be the twistor space for $\mathcal{M}$ and L can be taken to be the curve in $\widetilde{P\mathcal{T}}$ corresponding to the vertex of the light cone. This means that, according to the induced initial data from $\mathcal{M}$, $\mathcal{N}$ is foliated by α-planes and therefore the $\sim$'ed shear vanishes. We therefore see that, roughly speaking, half the data on $\mathcal{N}$ is lost. C. LeBrun has shown that it is always possible to blow down Q when $P\mathcal{T}$ is close to the hypersurface twistor space of a null cone in $\mathbb{M}$; the existence of a regular blowdown only requires conditions on the normal bundle of Q. In order to encode the extra data, one needs to also have the 'time' rate of change of the complex structure($\in H^1(P\mathcal{T},\Theta)$) as the hypersurface is evolved through the space-time. (This data will, of course, be subject to constraints.)

Null infinity: Null infinity suffers from the combination of the two above difficulties. Not only does the hypersurface twistor space fail to encode half the data, but also the second half of the data only appears as a holomorphic vector valued (0,1)-form, $\ddot{\bar\partial}$, which is the ***second*** derivative of the $\bar\partial$ operator as the hypersurface is evolved to second order into the space-time. This can be computed as follows. In the space-time with unphysical metric in which $\mathscr{I}$ is a finite null hypersurface with normal $\iota^{A'}\iota^{A}$, one can compute the evolution of the $\bar\partial$ operator to be:

$$\dot{\bar\partial}=\mathcal{L}_{\frac{z^{A'}\bar z^{A}}{|z\cdot\iota|^{-2}}\nabla_{AA'}}\bar\partial=\frac{\bar z_{A}\iota_{A'}dx^{AA'}}{(z\cdot\iota)^2(\bar z\cdot\bar\iota)}\Psi_{B'C'D'E'}z^{B'}z^{C'}z^{D'}\frac{\partial}{\partial z_{E'}}$$

Here the hypersurface twistor space at $\mathscr{I}$ is coordinatized by the coordinates, x^a of $\mathscr{I}$ itself and the spinor valued coordinate $z_{A'}$ up the fibre of the spinor bundle restricted to $\mathscr{I}$. Since Ψ vanishes on $\mathscr{I}$, $\dot{\bar{\partial}}$ vanishes and $\ddot{\bar{\partial}}$ has the identical form to the above expression with Ψ replaced by its first derivative, Ψ^0, in the direction transverse to $\mathscr{I}$ ($\nabla_{AA'}\Psi_{B'C'D'E'} = \iota_A \iota_{A'} \Psi^0_{B'C'D'E'}$). This expression is somewhat messy to write out in terms of the asymptotic shears and their integrals (it depends on integrals of non-linear combinations of σ and $\bar{\sigma}$). However, when $\bar{\sigma}=0$, (the 'googly' case) the hypersurface twistor space at $\mathscr{I}$ is ordinary flat twistor space, and $\ddot{\bar{\partial}}$ should be simple. However I havent yet worked it out as the details turn out to be slightly more problematic than expected. The cohomology class defined by $\ddot{\bar{\partial}}$ should vanish since hypersurface twistor spaces in self-dual space-times based on null cones are all flat. We should therefore have that $\ddot{\bar{\partial}} = \bar{\partial} V$ for some (1,0) vector field V.

A definition of positive frequency for gravitational initial data sets

One of the more striking results in twistor theory is the geometrization of the positive/negative frequency splitting for ZRM fields that is so important in quantum field theory. This might lead one to suggest (Mason 1989) that an initial data set should be said to be of positive frequency if the correponding hypersurface twistor space could be continued from a neighbourhood of PN with topology $S^3 \times \mathbb{R} \times S^2$ to a region with topology $\mathbb{R}^4 \times S^2$ which would be thought of as a deformed analogue of $\mathbb{PT}^+$ (so that one can fill in the $S^3 \times \mathbb{R}$ factor into a ball$\simeq \mathbb{R}^4$). (Cf the definition of a positive frequency non-linear graviton in Penrose 1976.)

One can check what this definition does in linearized theory by taking the expression for infinitesimal deformation of a hypersurface twistor space in flat space, $\mathbb{PT}$, due to a linearized solution of the field equations that I obtained in my D.Phil. thesis (see also my article in $\mathbb{TN}$ 20). The condition would require that the deformation be of positive frequency. This implies that the contribution from the ASD part of the field is of positive frequency since that appears directly in the formula. However, the expression for the infinitesimal deformation uses the reflection of the SD part of the field in the hypersurface. This reflection must be of positive frequency, so that the SD part of the field must be of negative frequency. As Abhay Ashtekar has pointed out, this is unfortunately unphysical, since it implies that the helicity of both the ASD and the SD contributions both have negative helicity, so that one has the tensor product of two −ve helicity graviton Fock spaces in linearized theory with this definition, instead of the product of the positive with the negative.

This definition may have some interest, since, firstly there are many Lorentzian real initial data sets which can satisfy this condition (at least in linearized theory), and for such initial data sets one will have a canonical twistorial definition of positive frequency for backround coupled ZRM fields.

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Some Stein Complementary Series for $GL(4,\mathbb{C})$

We are all familiar with the massless fields on M, $M^\pm$ and M^0. On $M^\pm$ they turn-out to be singular unitary representations of the group $U(2,2)$ (or $SU(2,2)$, if you prefer). Here the word 'singular' can be interpretted in a variety of ways. One version of the technical definition is to say that they do not contribute to the Plancherel measure for G. That is, when decomposing $L^2(G)$ (in analogy to the Fourier transform for $L^2(\mathbb{R})$) the singular representations don't appear. Alternatively, these are the representations which (almost) everybody missed when first trying to classify unitary representations.

The group $GL(4,\mathbb{C})$ has singular unitary representations of a type slightly different from the massless fields. These were first recognized by E. Stein. His lectures at Univ. of Notre Dame on his results make good reading. To write down these representations, proceed as follows:

As a homogeneous space for $G = GL(4,\mathbb{C})$, $M = G/P$ where

$$P = \left(\begin{array}{c|c} L_1 & B \\ \hline 0 & L_2 \end{array}\right).$$

Here, L_1, L_2, B and 0 are 2×2 blocks. The isotropy group P has a decomposition $P = MAN$ where

$$M = \left\{ \left(\begin{array}{c|c} L_1^0 & 0 \\ \hline 0 & L_2^0 \end{array}\right) \;\middle|\; |\det L_1^0| = 1 \text{ and } |\det L_2^0| = 1 \right\}$$

$$A = \left\{ \left(\begin{array}{c|c} \ell_1 I & \\ \hline & \ell_2 I \end{array}\right) \;\middle|\; \ell_1 \text{ and } \ell_2 \text{ are positive real \#'s} \right\}.$$

$$N = \left\{ \left(\begin{array}{c|c} I & B \\ \hline 0 & I \end{array}\right) \right\} \qquad (I = 2\times2 \text{ identity matrix.})$$

Thus, N is unipotent, A is abelian and M and A commute with each other. An irreducible representation of P is naturally a tensor product of irreducibles for M and A and the trivial representation of N. Via the standard principal bundle theory, each representation of P induces a homogeneous vector bundle on G/P. For the complementary series line bundles will suffice. The one-dimensional representations of P are its characters. The unitary characters of M are

$$\xi(m_1, m_2)\left(\begin{array}{c|c} L_1^0 & \\ \hline & L_2^0 \end{array}\right) = (\det L_1^0)^{m_1} (\det L_2^0)^{m_2}$$

where m_1 and m_2 are integers. Since A is abelian, all its irreducibles are one-dimensional:

$$\nu(s_1, s_2)\left(\begin{array}{c|c} l_1 I & \\ \hline & l_2 I \end{array}\right) = l_1^{s_1} l_2^{s_2} \qquad s_j = \sigma_j + i\tau_j .$$

These are unitary when $\sigma_1 = \sigma_2 = 0$. The resulting line bundle on M carries an invariant Hermitian inner product when ν is unitary. The representation of G on the L^2-sections is a 'unitarily induced' representation. They are well-known and not singular. However, the parameters s_j can move off the imaginary axis a little and still produce unitary representations. In the present case, "a little" means:

$$-2 < \sigma_1 < 2$$
$$-2 < \sigma_2 < 2 .$$

Now, however, the invariant inner product is gone since the inner product on the fibers is not invariant when $\sigma_j \neq 0$. To get the right Hermitian pairing, one constructs intertwining operators between the representation $\pi_{\xi,\nu}$ and its Hermitian dual. The Hermitian dual of a representation π on a Hilbert space is defined by

$$\pi^h(g) = \pi(g^{-1})^*$$

where $*$ means take the adjoint operator. To define the adjoint here, use the non-invariant inner product. Using an analytic continuation argument starting from the $\sigma_j = 0$ cases, it is possible to show that the intertwining operators, $A(s_1, s_2)$, are bounded, self-adjoint and invertible until one encounters poles at $\sigma_1 = \pm 2$, $\sigma_2 = \pm 2$. The operators for unitary ν are bounded, self-adjoint and invertible to begin with, since the representations of G are unitary, hence equivalent to their duals. The intertwining operators give new invariant inner products on the $\pi_{\xi,\nu}$ by defining

$$\langle \varphi, \psi \rangle^A = \langle \varphi, A\psi \rangle .$$

When the parameters s_1, s_2 reach the poles, the inner product will be either undefined or not positive definite.

Remarks:

① I don't know what these representations look like when restricted to $U(2,2)$. They will generally be reducible. For instance, they should contain copies of

$$\Gamma(\mathbb{M}^{\pm}, \mathcal{O}[m_1][m_2]').$$

One hope would be that they contain a sequence of interesting representations in some fashion similar to the way massless field representations come from restricting the metaplectic representation to $U(2,2)$.

② When forming the line bundle on $\mathbb{M}$ one gets a holomorphic bundle only when the starting representation is holomorphic. The representations used here look like:

$$\left(\begin{array}{c|c} L_1 & \\ \hline & L_2 \end{array}\right) \longmapsto \frac{(\det L_1)^{m_1}(\det L_2)^{m_2}}{|\det L_1|^{s_1-m_1}|\det L_2|^{s_2-m_2}},$$

which is not holomorphic for $s_1 \neq m_1$, $s_2 \neq m_2$.

Ed Dunne.

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Abstract: (To appear in Physics Letters A)

Nonlinear Schrödinger and Korteweg-de Vries are Reductions of Self-Dual Yang-Mills

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Abstract

The non-linear Schrödinger (NS) and KdV equations are shown to be reductions of the self-dual Yang-Mills (SDYM) equations. A correspondence between solutions of the NS and KdV equations and certain holomorphic vector bundles on a complex line bundle over the Riemann sphere is derived from Ward's SDYM twistor correspondence. Remarkably the twistor correspondence generalizes to the NS and KdV hierarchies when complex line bundles of higher Chern class are used. We discuss solitons and inverse scattering.

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Double box diagrams

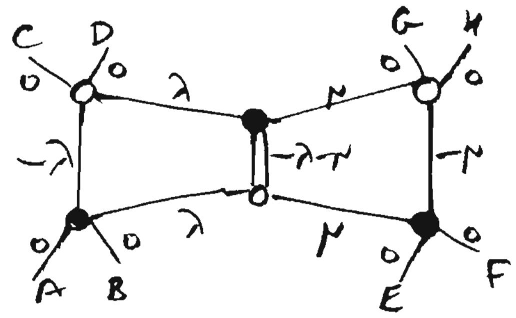
Introduction: The attempt [TN 27] to perform the direct translation of Feynman propagators and vertices in general position led to the writing down of large twistor diagrams too difficult to evaluate at present. These large diagrams essentially arise as compositions of simpler "box" diagrams. It therefore looks useful to approach general diagram-bulding by a detailed study of the simplest such composition, namely the double box. There are several other reasons for studying it:

(1) the early study of the double box by RP and George Sparling [PhD thesis, 1974] has remained uncompleted.

(2) it turns out that new light is cast on the crossing symmetry problem for the *single* box.

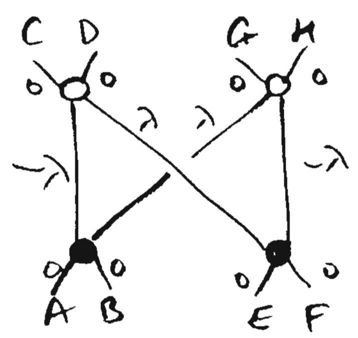
(3) there are applications to electromagnetic and SU(2) amplitudes of particular interest.

The problem: Sparling's approach. In what follows we shall use only the original projective diagram calculus. To begin with we can confine ourselves to *scalar* (elementary) external states, so that the most general double box diagram we need consider is

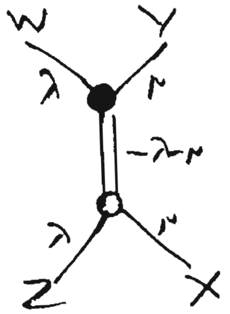


Here λ and $\mu \in \mathbb{C}$ cannot be non-zero integers; (but we shall be particularly interested in studying limits as they approach integer values.)

We now follow Sparling's program for an explicit evaluation. The idea of this approach is that we first integrate out U_α and V^α, reducing the integral to one which we can treat by the methods used for the single box diagram



To do this we first note that there is a contour for



allowing $Z^{\alpha} = X^{\alpha}$, $W_{\alpha} = Y_{\alpha}$, the result of the integral being

$$\Gamma(1+\lambda)\Gamma(1+\mu)\,(W.Z)^{-1-\lambda}(Y.X)^{-\mu-1}\, I_p\!\left(\lambda+1,\mu+1;\ \frac{W.Z\;Y.X}{Y.Z\;W.X}\right)$$

where $I_p(\lambda+1, \mu+1; u)$

is the hypergeometric function defined by $\int_0^\infty dz\, z^{\mu}(1+zu^{-1})^{-1-\lambda}(1+z)^{-1-\mu}$

Sparling left the problem at essentially this point. To pursue it to a conclusion we have to analyse the "period" contours for the UV integral, i.e. contours corresponding to the period structure of the hypergeometric function. The key point is that only a very particular choice of contour will yield a result which is actually an ***amplitude***: i.e. a mapping from in-states and out-states to **C**. The choice is such that in the result of the UV integral,

$I_p(u)$ is replaced by $I_H(u) = \{(1-e^{2\pi i\lambda})+(1-e^{2\pi i\mu})\}\, I_p(u) + I_c(u)$

where $I_c(u)$ is $\oint_{\substack{\text{round}\\ [-1,0]}} dz\, z^{\mu}(1+zu^{-1})^{-1-\lambda}(1+z)^{-1-\mu}$

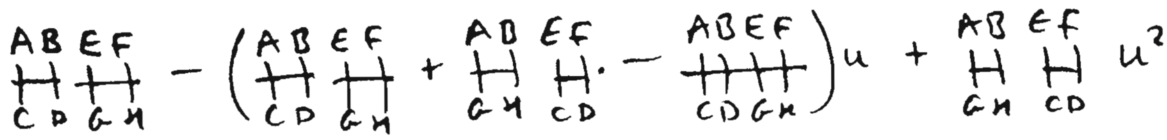
The defining feature of this peculiar combination is that the *period* of $I_H(u)$

about the branch point at $u = 0$ is $(1-e^{2\pi i\lambda})(1-e^{2\pi i\mu})\, I_p(u)$

Explicit contours can be constructed for the WZ, then the YX integrals, essentially in analogy with the integration of the single box, although this requires care. The final result is, in closed form

$$\frac{I_p(\lambda,\mu;r_1) - I_p(\lambda,\mu;r_2)}{r_1 - r_2} = \frac{1}{2\pi i}\oint \frac{I_p(\lambda,\mu;u)}{Q(u)}\, du$$

where $Q(u)$ is the standard quadratic



formed from the four external states, and r_1, r_2 are its roots. Note that

$$I_p(\lambda,\mu;u)$$

is analytic at $u = 1$. This is the feature which ensures that we now have a genuine amplitude for AB, EF corresponding to in-states; CD, GH to out-states. Had we chosen another contour at the first stage we should still obtain a finite integral (at least for AB ... GH in general position) but without this essential physical property.

The result can be rewritten as the non-compact integral $\displaystyle\int_0^\infty \frac{I_c(\lambda,\mu;u)\,du}{Q(u)}$

where $\displaystyle I_c(\lambda,\mu;u) = \oint_{\substack{\text{round}\\ [-1,0]}} dz\, z^{\mu-1}(1+zu^{-1})^{-\lambda}(1+z)^{-\mu}$

the representation being valid provided $\operatorname{Re}\lambda > -1,\ \operatorname{Re}\mu > -1$

in analogy with $\displaystyle\int_0^\infty \frac{u^\lambda\,du}{Q(u)}$ $\quad(-1 < \operatorname{Re}\lambda < 1)$

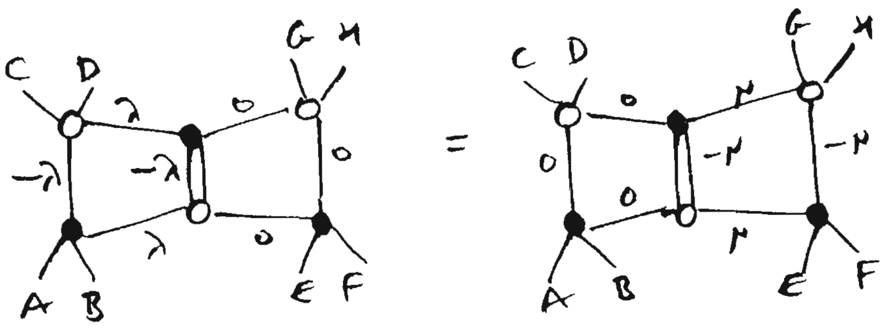
for the single box.

Deductions from the solution:

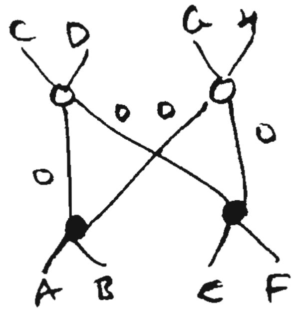
This allows us to note at once some special cases:

(i): $\lambda = 0$ or $\mu = 0 \implies I_c(\lambda,\mu;u) \equiv 1$

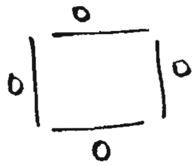
hence



$$= \int_0^\infty \frac{du}{Q(u)} =$$



This is consistent with

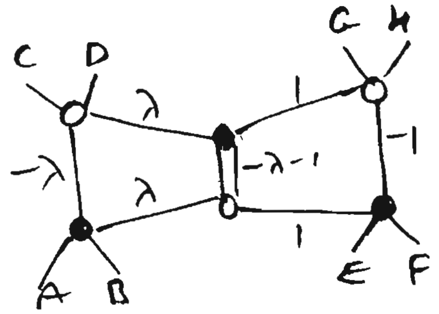


acting as a two-twistor projection operator for eigenstates of spin 0.

(ii) considering the limits

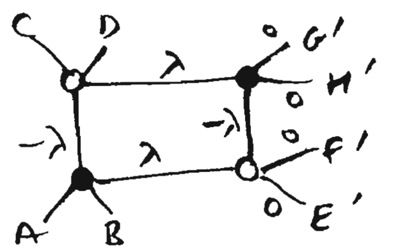
$$\mu \to 1: \quad I_c(\lambda,\mu;u) \to (1-u^{-1})^{-\lambda}$$
$$\lambda \to 1: \quad I_c(\lambda,\mu;u) \to (1-u^{-1})^{-\mu}$$

we find



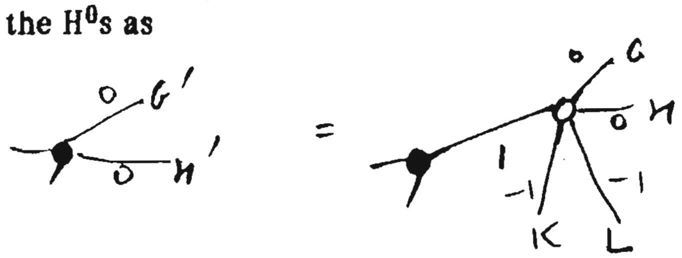
$$= \int_0^\infty \frac{(1-u^{-1})^{-\lambda}}{Q(u)}\,du$$

which one may check is just the amplitude previously obtained by taking a so-called "hard contour" for

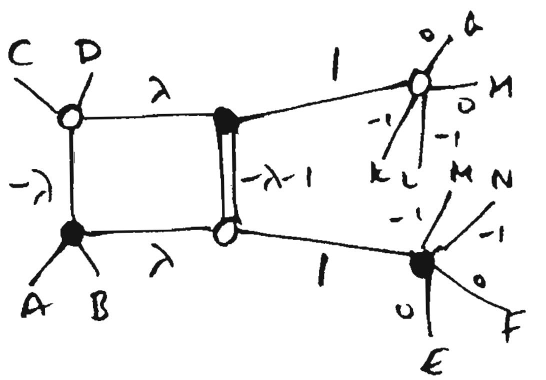


$(GH = G'H', \;\; EF = E'F')$

This means that the *double* box (with $\mu = 1$) supplies a genuine contour integral involving genuine H^1s for the "missing" channel of the *single* box. Note that the "hard contour" so-miscalled, is one which treats two of the external H^1s as if they were H^0s. Such a contour can better be described by writing the H^0s as

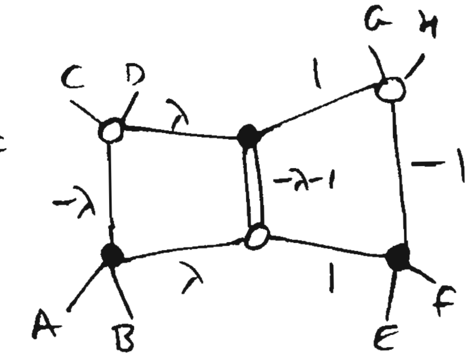


so that the external state is genuinely an H^1 and parameter-dependence is explicitly given by the K, L . The old results on "hard contour" integration (APH *Physica* **114A**, 157 (1982)) can now be recast as the statement that

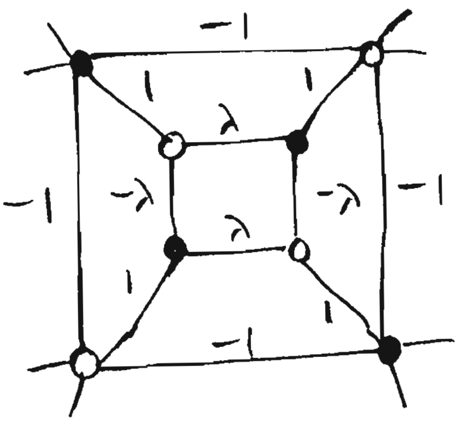


$$= \int_0^\infty \frac{(1-u^{-1})^\lambda \, du}{Q(u)}$$

$$=$$



By an extension of this observation we see that larger diagrams e.g.



contain within them all channels for the single box. This provides a new view of the crossing symmetry problem - closely related to the ideas for extending the simple box discussed in TN 25.

A different construction for the contour

Having derived a meaningful formula for the result of composing two boxes, one can ask whether it could be arrived at more naturally. In fact there are probably several ways: here I will just remark on a contour construction which works the opposite way to Sparling's. Take the special case of coincident in- and out-states, i.e. AB = EF, CD = GH. Then we may first integrate out the *external* vertices and obtain:

$$\oint \frac{DUV\ \Gamma(2-\lambda-\mu)\Gamma(1+\lambda)\Gamma(1+\mu)\begin{pmatrix} A\ B \\ C\ D\end{pmatrix}^{\lambda+\mu}}{\begin{pmatrix} U \\ V\end{pmatrix}^{2-\lambda-\mu}\begin{pmatrix} U\ A\ B \\ V\ C\ D\end{pmatrix}^{1+\lambda+\mu}}$$

One may give explicit coordinates for U_α and V^α which diagonalise the two bilinear forms which appear, and in these coordinates it is fairly straightforward to define a contour in UV space over which the integration yields

$$\frac{\Gamma(1+\lambda)\Gamma(1+\mu)}{\Gamma(1+\lambda+\mu)}\begin{pmatrix} A\ B \\ C\ D\end{pmatrix}^{-2}$$

This is in agreement with the formula $\begin{pmatrix} A\ B \\ C\ D\end{pmatrix}^{-2}\displaystyle\int_0^\infty \frac{I_c(\lambda,\mu;u)\,du}{(1-u)^2}$

which we would obtain in this special case from the contour as constructed in the Sparling way. It seems clear (but not proved) that the different constructions do represent the same homology class. Using the second construction it is hard to generalise to AB, CD, EF, GH in general position. But it has the advantage of showing explicitly that the limits $\lambda\to 1$, $\mu\to 1$ may be regarded as corresponding to taking a contour-with-boundary.

Other channels for the double box?

Putting the result of the integration in the form $\displaystyle\int_0^\infty \frac{f(u)\,du}{Q(u)}$

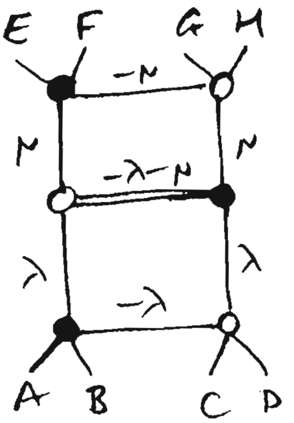
is particularly useful because it is known that this is just the amplitude which corresponds to the insertion of a ***momentum-space*** kernel of form

$$f\left(\frac{k_1\cdot k_4}{k_1\cdot k_2}\right)$$

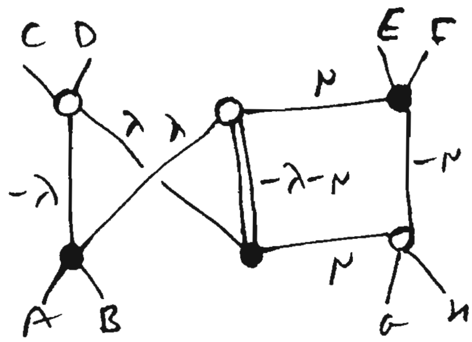
in the appropriate channel. It follows immediately that the amplitudes in the other channels must be:

$$\int_1^\infty \frac{I_c(\lambda,\mu;u)}{Q(u)}\,du, \qquad \int_0^1 \frac{I_c(\lambda,\mu;u)}{Q(u)}\,du$$

i.e. these formulae must in some sense correspond with

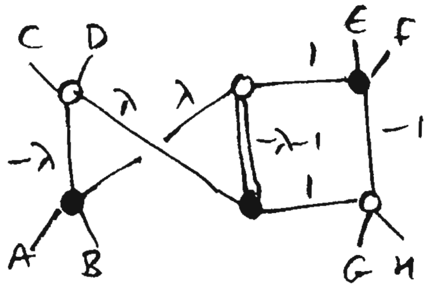


,



respectively. The first of these obviously cannot be a genuine contour integral as it stands, but there does seem to be a "hard contour" (in the sense explained above) which effects this correspondence. It is an open question as to whether there is a contour integral for the second of these crossed channels. If there is one, it certainly can not be obtained by integrating out the inner variables first (the required amplitude, regarded as an analytic function in λ and μ, has a pole at $\lambda+\mu = 2$; but this pole is cancelled as soon as we do the UV integral in the Sparling manner.)

Looking at the limiting case $\mu \to 1$, we note that the answers for the integrals



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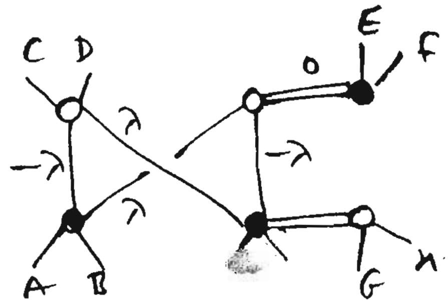
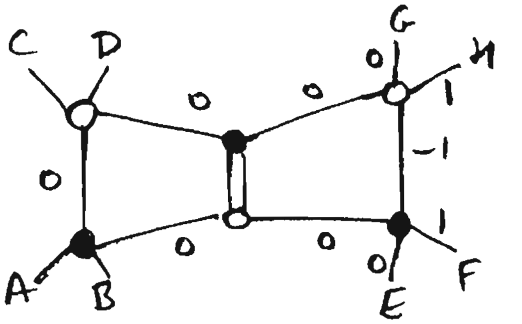


must agree, and at first sight it would appear that the contour for the double-box diagram, if it exists, must be identifiable in this limit with the contour for the twistor-transformed single-box. However, this turns out to be incorrect (see below).

Extension to external states of non-zero helicity

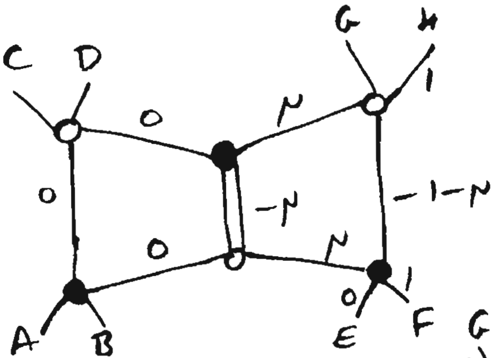
It is fairly straightforward to use spin-raising and integrating-by-parts techniques to generalise these results to non-scalar states. The first example shows that the results are not always what might be expected (by me, anyway):



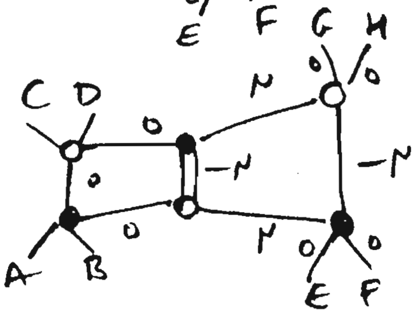
$$\equiv 0$$

Proof: this is

$$\lim_{\mu \to 0}$$



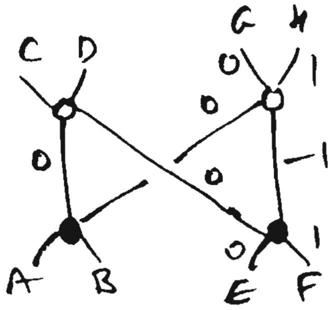
$$= \lim_{\mu \to 0} \frac{1}{\mu} \frac{\partial}{\partial F} \cdot \frac{\partial}{\partial H}$$



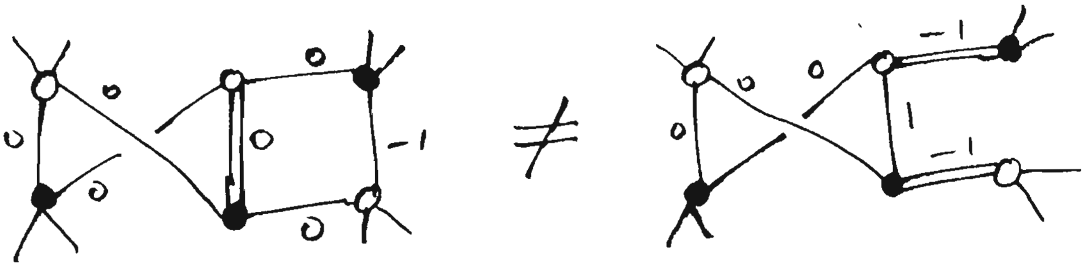
$$= \lim_{\mu \to 0} \frac{1}{\mu} \frac{\partial}{\partial F} \cdot \frac{\partial}{\partial H} \int_0^\infty \frac{du}{Q(u)}$$

$$= \lim \frac{1}{\mu} \cdot 0 \qquad = 0.$$

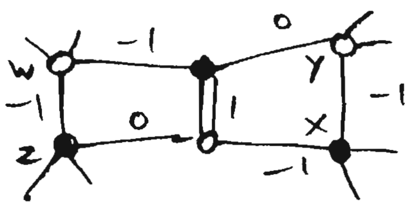
One might have expected this to be



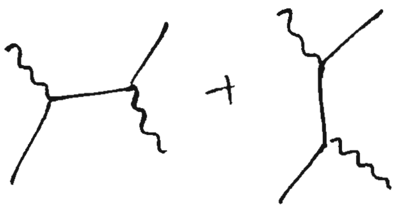
This means that if a contour exists for the crossed channel, we shall have



the RHS diagram being non-zero. Hence the agreement of a double box in its limiting case, and the twistor-transformed single box, will not hold in general for non-zero external helicities. Acordingly, it requires careful checking to ensure the validity of the double box originally written down by RP for Compton scattering:

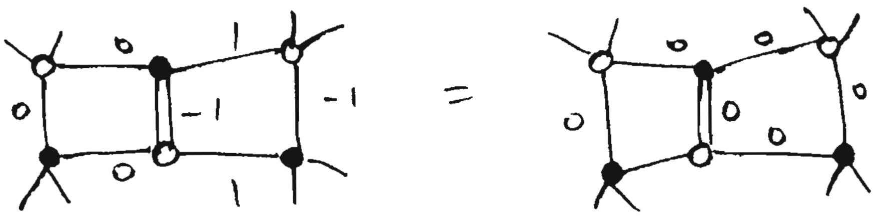


for

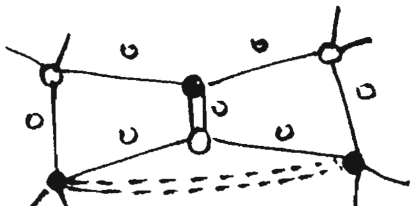


In fact this follows by operating on the double box with $\partial_W \partial_Y \quad Z \partial_W$

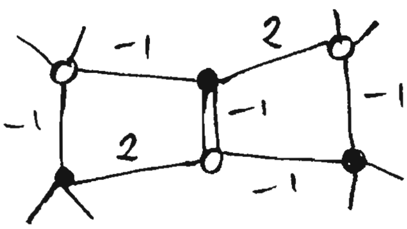
We obtain three terms, two of which cancel by virtue of the identity



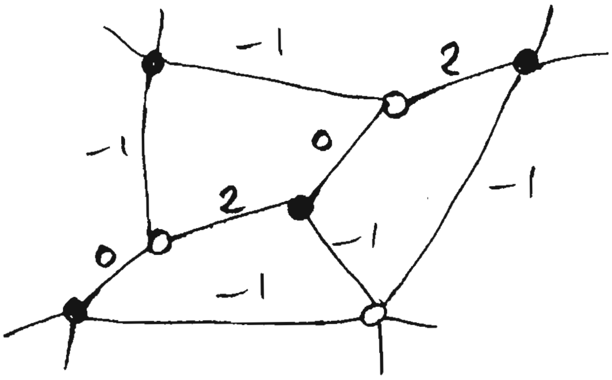
and leave



This is equivalent to agreement with the Feynman calculation. Similarly we can show that

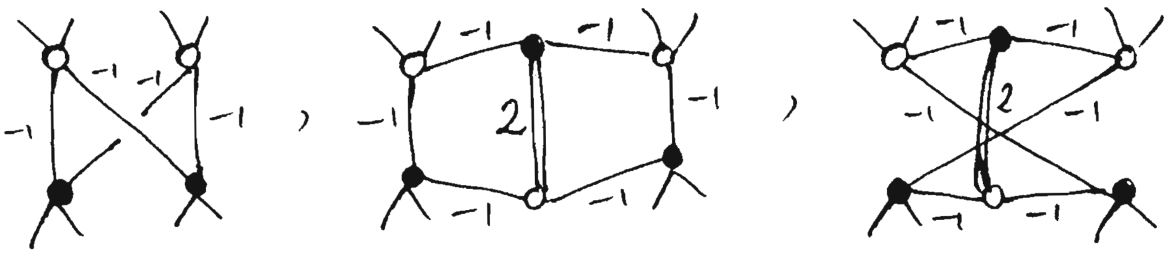


is also valid (for the appropriate channel). Hence the extended diagram



does contain within it the contours for all channels.

In a similar way we can use the double box integral to represent the "missing" channels for other first-order interactions. At this point we shall assume that the infra-red divergences may be consistently be removed by application of the inhomogeneous propagators within the double-box integral. Then of particular interest are amplitudes for SU(2) processes (see TN 23). We find for instance that the representation of pure gauge field self-interaction as a summation over



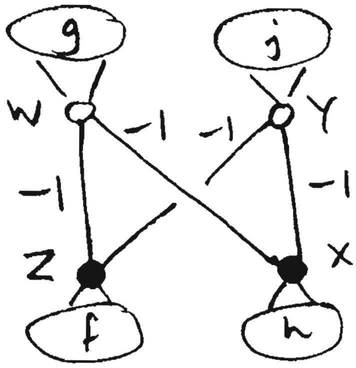
is fully justified (in this channel).

Andrew Hodges

Pochhammer contours in twistor diagrams

Pochhammer contours have a role in convolving the anti-derivative twistor diagram elements in non-projective twistor space. The technicalities may be of interest in view of RP's suggestions regarding holomorphic linking. In this note I mostly consider the single box with its one channel, but the ideas naturally extend to the double box.

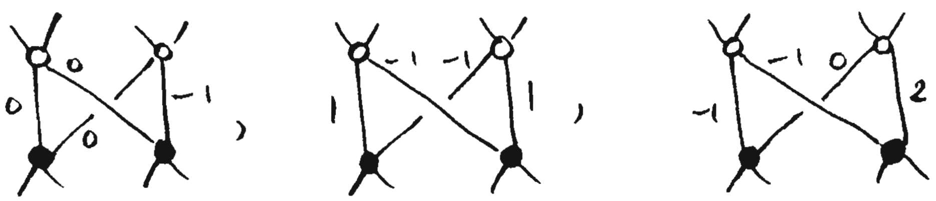
The need for Pochhammer contours is seen most directly by considering the twistor diagram:



where f, g, h, j are of homogeneity (-4)

(1)

This occurs in the interaction of SU(2) gauge fields [TN 23] and so is of interest in its own right; but it is just as important to see it as the *most general* case of a single-box diagram with integer helicities, from which all other cases such as

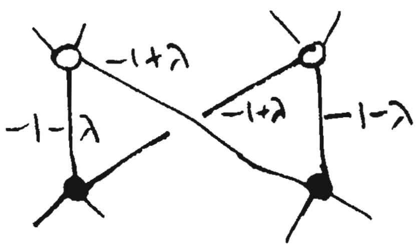


etc.

can be derived.

Firstly, consider within the projective diagram calculus the limit

$\lim_{\lambda \to 0}$

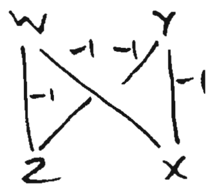
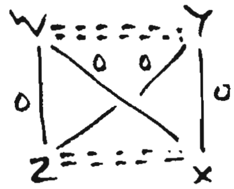


(2)

It is infinite, indicating (as is to be expected) infra-red divergence. If we replace the projective diagram calculus by the inhomogeneous propagators [APH, *Proc. R. Soc. Lond.* **A397** 341-374 (1985)] we obtain a new interpretation of (1) as the integral:

$$\oint f(Z^\alpha)g(W_\alpha)h(X^\alpha)j(Y_\alpha)\{\gamma+\log(W.Z-k_1)\}\{\gamma+\log(W.X-k_2)\}$$
$$\{\gamma+\log(Y.Z-k_3)\}\{\gamma+\log(Y.X-k_4)\}\, D^4ZWXY \qquad (3)$$

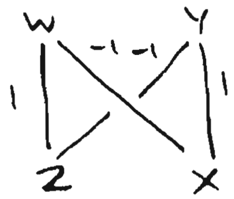
An essential point here is that we must preserve the condition

$$(W\partial_Z\, Z\partial_W)(W\partial_X\, X\partial_W)$$  $=$  (4)

Thus any proposed contour for the new integral must yield the correct "delta-function" answer when applied to the RHS diagram in (4). Similarly,

by applying $(W\partial_X\, X\partial_W)^2$

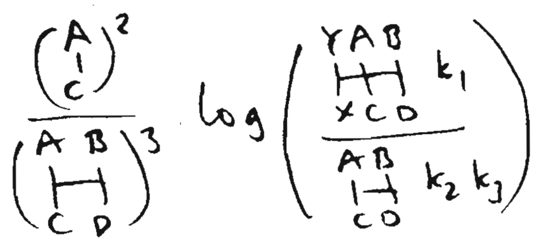
we see that the contour must give a consistent result when applied to the inhomogeneous "Møller scattering" diagram



We use this necessary condition as a guideline on how to proceed, trying to construct a contour for (3) which will meet this condition.

Let f, g, h, j be elementary, so $f(Z^\alpha)=\dfrac{2}{A.Z(B.Z)^3}$, $g(W_\alpha)=\dfrac{2}{W.C(W.D)^3}$ etc.

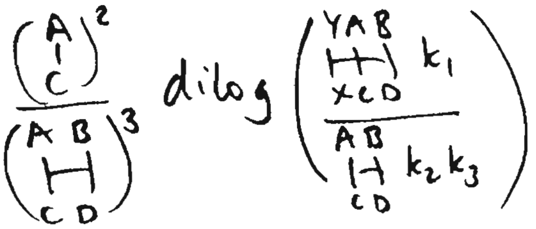
The Z^α and W_α integration can indeed be done by analogy with the "Møller" diagram, the result being essentially



which then, pursuing the analogy, should be combined with the remaining

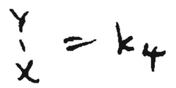
$\{\gamma+\log(Y.X-k_4)\}$ factor. In the Møller case, this factor is a double pole which could be surrounded by an S^1 . But now we have a *branch point* instead of a pole and there is in fact *no* contour corresponding to this S^1.

The Pochhammer contour saves the situation. However, to use it we have to back-track and first do the WZ integration differently, abandoning temporarily our Møller guideline. We use a contour which allows $Y_\alpha = 0$, $X^\alpha = 0$ [the computation may quite conveniently be done by expanding the WX and YZ factors in inverse powers of k]. The result of the WZ integration is then (essentially)

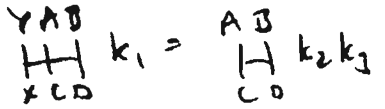


$$\left[\mathrm{dilog}\, z = -\int_0^z \log(1-t)\frac{dt}{t}\right]$$

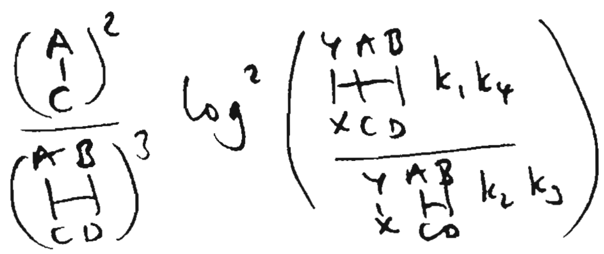
i.e. something whose *period* is the logarithmic expression we had before. Now this dilogarithm, and the remaining logarithmic factor, *can* be successfully combined by performing a Pochhammer contour integration around the branch point at



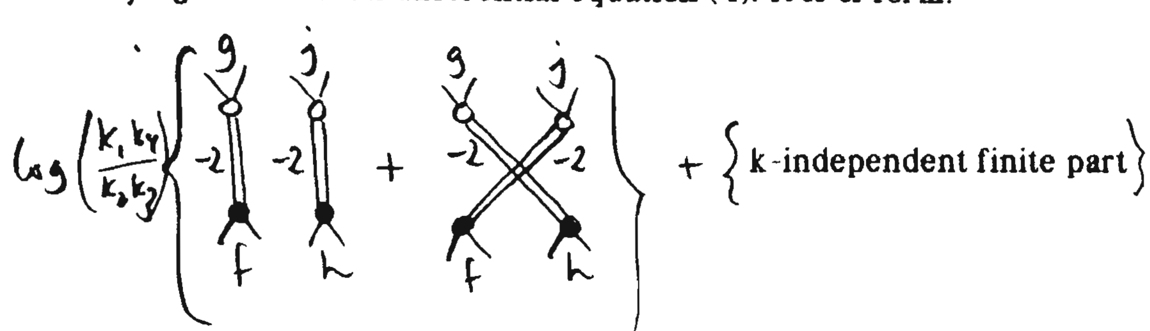
and the newly introduced branch point at



The result of this is (essentially) equivalent to the *projective* twistor factor



and the remaining integration over Y and X yields a finite answer satisfying the essential differential equation (4). It is of form:



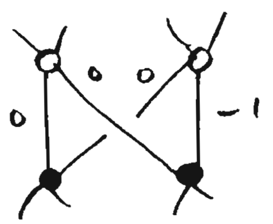
where this k-independent part is the same as we should have got from the projective twistor integral

$$\oint DWZXY \ \frac{1}{4!}\log^4\left(\frac{W\cdot Z \ Y\cdot X}{W\cdot X \ Y\cdot Z}\right) f(Z^\alpha)g(W_\alpha)h(X^\alpha)j(Y_\alpha)$$

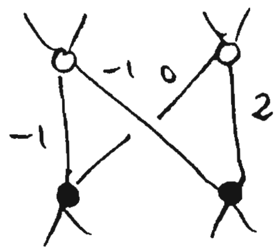
which is itself the natural regularisation of the divergent limit (2). We must of course check that the new contour thus constructed is in fact one that could validly have been employed for the Møller diagram or for the RHS in (4); indeed this is easy to check by observing how the Pochhammer integral reduces to the residue calculus when one of the branch points happens to reduce to a pole.

Further remarks:

1. Although this construction has been motivated by requiring an extension of the diagram calculus to the (-1,-1,-1,-1) diagram, it should not be thought of as specific to this diagram. In fact, it is actually more consistent to use this construction in *all* box diagrams, since it has the effect of putting all the lines on an equal footing. This comment applies equally to diagrams like

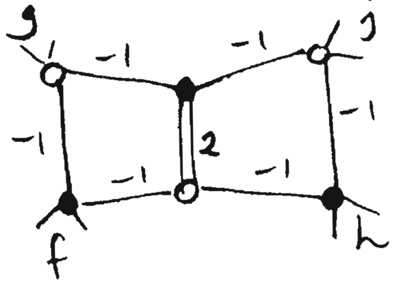


,



which are *not* infra-red divergent.

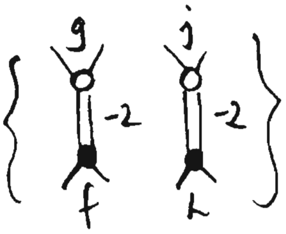
2. Subject to some detailed checking and computation, it should be straightforward to combine this construction with the double-box analysis and so give a finite evaluation of



and hence of the complete SU(2) interaction [TN 23].

3. There is more freedom of choice for the contour in (3) than has been indicated above. The contour described is one in which the logarithmic factors play a role only through defining branch points (equivalently, the Euler constant γ plays no role.) But we do have the freedom to add on pieces of contour which "see" the logarithm and contribute terms like

$$(\gamma + \log k_2)$$

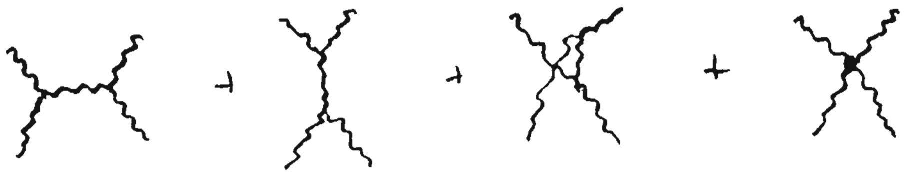


(Contours *do* play this role in the integrals which generate mass eigenstates.) However, one point of interest in the contour as originally described above is that it seems very likely to be equivalent to a contour-with-boundary construction; i.e. that the amplitude could be rewritten as

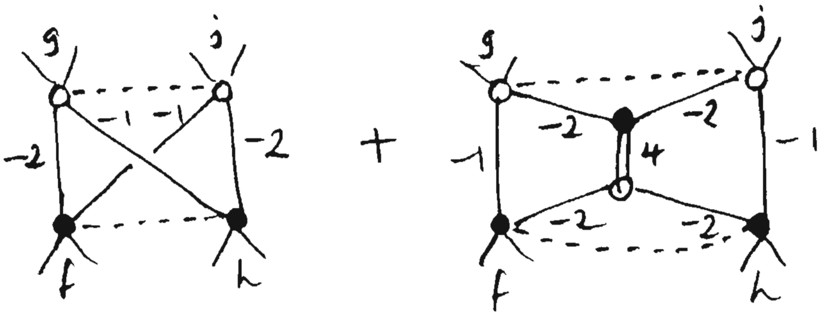
$$\oint_{W\cdot Z = k_1,\ W\cdot X = k_2,\ Y\cdot Z = k_3,\ Y\cdot X = k_4} f(Z^\alpha)\, g(W_\alpha)\, h(X^\alpha)\, j(Y_\alpha) \quad D^4 ZWYX$$

This accentuates the intriguing similarity to the *open string* calculation (TN 27)

4. The diagrams considered here are also particularly relevant to *graviton-graviton* scattering; the Feynman sum



can be written (for helicity eigenstates) as the sum of the twistor diagrams



which have the same singularity structure as the (-1,-1,-1,-1) box diagrams discussed above.

Andrew Hodges

Twistors and the Regge Calculus

Regge calculus has been popular for many years as a method of reducing the problem of describing curved space-times to piecewise linear combinatorics. A substantial difficulty is that piecewise linear space-times satisfying Regge's field equations do not satisfy the exact Einstein vacuum equations. Indeed, the connection between the vacuum equations and solutions of Regge's equations is far from direct requiring sophisticated limiting arguments (see discussions in papers by J. Barrett etc. in C.Q.G. in the mid '80's). In this note I would like to point out that Regge space-times can indeed satisfy the Einstein vacuum equations, but only when the signature of the metric is (2,2) and the 'bones' are α- or β-planes.

The piecewise linear space-times that Regge considered have their curvature supported on codimension 2 submanifolds. The curvature is entirely due to conical singularities around these so called 'bones'. The curvature can readily be seen to have the form:

$$R_{abcd}=\epsilon\delta^2 B_{ab}B_{cd}$$

where $B_{ab}=B_{[ab]}$ is the 2-form orthogonal to the 'bone' (so that in particular B_{ab} is simple), ϵ is related to the angle deficit around the bone, and δ^2 is the δ-function supported on the bone. The Ricci curvature is thus: $$R_{ab}=\epsilon\delta^2 B_a{}^c B_{bc}$$
For $R_{ab}=0$ we must therefore have that B_{ab} is a null 2-form: $B_{ab}=o_A o_B \varepsilon_{A'B'}$ or $o_{A'}o_{B'}\varepsilon_{AB}$ for some o_A or $o_{A'}$. Thus we see that the 'bone' is a β-plane or an α-plane respectively. This can only be real when the space-time has signature (2,2). (These ideas would be difficult to make sense of in the complex because of the use of δ-functions: one would perhaps need to develop some kind of hyper-functional interpretation.)

Example: Probably the simplest example of such a space-time is obtained by choosing a (real) β-plane through the origin aligned along o^A, and then identifying the space-time with itself dragged along the integral curves of the anti-self dual killing vector $K=x_B{}^{A'}o^{(B}o^{A)}\partial_{AA'}$.

To be more precise, choose coordinates $u^{A'}=x^{AA'}o_A$ and $x^{A'}=x^{AA'}\iota_A$ so that the metric is given by:

$$ds^2=\varepsilon_{A'B'}du^{A'}dx^{B'} \qquad \text{and} \qquad K=u^{A'}\partial/\partial x^{A'}$$

and the β-plane is given by $u^{A'}=0$. Then introduce the coordinates $\rho=u_{A'}x^{A'}$ and $\lambda=(\phi_{A'B'}u^{A'}x^{B'})/(\phi_{A'B'}u^{A'}u^{B'})$ where $\phi_{A'B'}=o_{A'}o_{B'}+\iota_{A'}\iota_{B'}$ so that $(u^{A'},\rho,\lambda)\to(u^{A'},\rho,\lambda+\epsilon)$ is the isometry generated by K. Then, we can identify λ with $\lambda+a$ for some a. (That is we can identify $x^{A'}$ with $x^{A'}+au^{A'}$ for some a.) The resulting space-time has a more interesting topology than is usually associate with 'cosmic string' type solutions (presumably $S^1\times S^1\times\mathbf{R}^2$). The holonomy around the S^1 factor caused by the identification can be checked to be $\delta_A{}^B+ao_Ao^B$

Interesting questions are:
1) What is the non-linear graviton construction for this space-time?
2) Is it possible a) to have two such bones meeting in a point, presumably there are consistency conditions. b) Is it possible to incorporate both S.D. and A.S.D Weyl curvature in a nontrivial way (i.e. with an A.S.D. bone intersecting an S.D. bone nontrivially)? It seems likely that there is no difficulty when the ASD bones miss the SD bones as will generically be the case.

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