A Twistorial Approach to the Full Vacuum Equations

In an attempt to explore possible connections between Ashtekar variables and hypersurface twistors, we have been led to a (tentative) new approach to the twistorial description of general relativity.

Let $\mathcal{M}$ be a complexified (analytic) space-time, and $\mathcal{H}$, a complexified (analytic) spacelike hypersurface in $\mathcal{M}$. A point in the projective hypersurface twistor space $\mathbb{P}\mathcal{T} = \mathbb{P}\mathcal{T}(\mathcal{H})$ is represented as an α-curve in $\mathcal{H}$. An α-curve is a (complex) null curve in $\mathcal{H}$ with tangent vector of the form $\tilde{\pi}^A \pi^{A'}$, where

$$\tilde{\pi}_A = t_A{}^{A'} \pi_{A'},$$

the vector t^a being normal to $\mathcal{H}$ (and can be normalized, if so desired, by

$$t_a t^a = 2, \quad \text{so that} \quad t_A{}^{B'} t_{B'}{}^{C} = -\varepsilon_A{}^C \;\&\; t_{A'}{}^{B} t_B{}^{C'} = \varepsilon_{A'}{}^{C'}$$

whence $\tilde{\tilde{\pi}}_{A'} = -\pi_{A'}$, with $\tilde{\lambda}_{A'} = t_{A'}{}^{A}\lambda_A$ etc.), and where we require

$$\tilde{\pi}^A \pi^{A'} \nabla_{AA'} \pi_{B'} = 0. \quad \ldots\ldots \text{(A)}$$

The non-projective hypersurface twistor space $\mathcal{T} = \mathcal{T}(\mathcal{H})$ is a line bundle over $\mathbb{P}\mathcal{T}$, with fibres given by the scalings for $\pi_{A'}$.
(See Spinors & Space-Time, vol. 2 pp. 215-218, R.P. & W.R.)

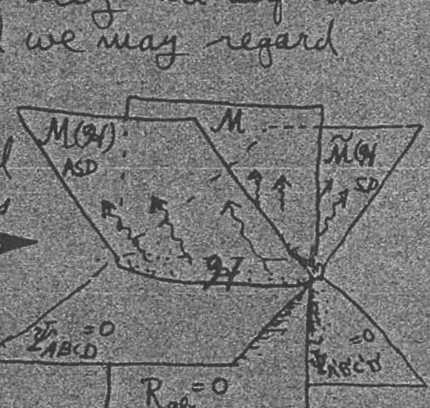
As a point of contact with Ashtekar's variables, note that the $\nabla_{AA'}$ in (A) is his (primed) spin connection — though only the conformal part of it is being used here — and his soldering form is implicitly involved in $\tilde{\pi}^A \pi^{A'} \nabla_{AA'}$. To obtain the entire (primed) spin connection we should need to incorporate

$$\tilde{\pi}^A \pi^{A'} \nabla_{AA'} \nu_{B'} \qquad \text{with } \nu_{B'} \text{ independent of } \pi_{B'}. \ \ldots \text{(B)}$$

A helpful concept, in relation to hypersurface twistors is that of "Heaven on Earth" $\mathcal{M}(\mathcal{H})$, which is the "non-linear graviton" construction applied to $\mathbb{P}\mathcal{T}$ (or $\mathcal{T}$); i.e. we take holomorphic curves in $\mathbb{P}\mathcal{T}$ (homologous to those representing points in $\mathcal{H}$) to represent points in $\mathcal{M}(\mathcal{H})$. Note that $\mathcal{M}(\mathcal{H})$ is conformally anti-self-dual (but not conformal to vacuum, in general), and we may regard

$$\mathcal{H} \subset \mathcal{M}(\mathcal{H}).$$

(We could define $\tilde{\mathcal{M}}(\mathcal{H})$ similarly, as a self-dual space, where primed and unprimed spinor indices are interchanged.) The picture



may be helpful. If $\mathcal{M}$ is vacuum, we can regard $\mathcal{M}$ as the Einstein evolution of $\mathcal{H}$, whereas $\mathcal{M}(\mathcal{H})$ is a conformal manifold evolved from the (conformally invariant part of) the same data at $\mathcal{H}$, but evolved according to $\tilde{\Psi}_{A'B'C'D'} = 0$. The space $\mathcal{M}(\mathcal{H})$ is "equivalent information" to $\mathcal{T}$, and it provides a useful way of studying the global structure of $\mathcal{T}$. (Similarly, the conformally self-dual space $\tilde{\mathcal{M}}(\mathcal{H})$, evolved from $\mathcal{H}$ according to $\Psi_{ABCD}=0$, gives equivalent information to the "dual" hypersurface twistor space defined by β-curves (compl. conj. of Ⓐ) on $\mathcal{H}$.)

The complex conformal space $\mathcal{M}(\mathcal{H})$ provides a good part of the initial data needed at $\mathcal{H}$ for the evolution of Einstein's equations (to give $\mathcal{M}$). However, two additional ingredients are needed:

① a conformal scale (equivalent to Ⓑ above)
② the location of $\mathcal{H}$ itself, within $\mathcal{M}(\mathcal{H})$.

Here we make use of an idea (cf. L.J.M. in TN 28: "Hypersurface Twistors"; C.R.LeB. TN 12 "Metrics as cohomology classes") whereby ① and ② are coded as two solutions of the conformally invariant wave equation on $\mathcal{M}(\mathcal{H})$

$$(\Box + \tfrac{R}{6})\Omega = 0, \qquad (\Box + \tfrac{R}{6})\chi = 0.$$

Using $g_{ab} \mapsto \Omega^2 g_{ab}$, we scale $\Omega \mapsto \Omega^{-1}\Omega = 1$ (so that R is scaled to zero) and $\chi \mapsto \Omega^{-1}\chi$, and we use:

$$\chi = 0 \quad \text{to define the location of } \mathcal{H}.$$

The freedom in Ω is precisely the freedom in choosing a conformal scale on $\mathcal{M}(\mathcal{H})$ so that $R=0$; the freedom in χ is the freedom in the choice of $\mathcal{H}$ within $\mathcal{M}(\mathcal{H})$ together with the choice of "lapse function", provided by the choice of normal (where now t_a need not be normalized by $t_a t^a = 2$)

$$t_a = \nabla_a \chi \qquad \text{at } \mathcal{H}$$

in $\mathcal{M}(\mathcal{H})$ — or, equivalently, in $\mathcal{M}$. (Note that $\chi=0$ at $\mathcal{H}$, together with $\nabla_a\chi$, give initial data for $(\Box+\frac{R}{6})\chi=0$ in $\mathcal{M}(\mathcal{H})$.)

We could specify Ω and χ as two elements of $H^1(\mathcal{O}(-2))$ on $\mathbb{P}\mathcal{T}$ — but we prefer to use a <u>bundle</u> $\mathbb{B}$ over $\mathbb{P}\mathcal{T}$ (or $\mathcal{T}$). We take $\mathbb{B}$ as an affine rank 2 bundle whose fibres are solutions of

$$\nabla_A \nu_{A'} = \nabla_{AA'}\chi \qquad \text{on an } \alpha\text{-plane in } \mathcal{M}(\mathcal{H}) \ldots Ⓒ$$

(Note that, being ASD, $\mathcal{M}(\mathcal{H})$ has α-planes — which meet $\mathcal{H}$ in the α-curves — and each α-plane corresponds to a point of $\mathbb{P}\mathcal{T}$.) Here we define

$$\nabla_A = \pi^{A'}\nabla_{AA'} \quad \{\text{derivative operator in } \mathcal{M}(\mathcal{H})\},$$

which differentiates within the α-plane, and we assign a homogeneity -1 to $\nu_{A'}$, with respect to $\pi_{A'}$. (Thus, under $\pi_{A'} \mapsto \xi\pi_{A'}$,

we take $\nu_{A'} \mapsto \zeta^{-1}\nu_{A'}$.) Note that χ has homogeneity 0.
Observe that since $\tilde{\Psi}_{A'B'C'D'} = 0$ in $\mathcal{M}(\mathcal{H})$, we have

$$\nabla_A \nabla_B \nu_{C'} - \nabla_B \nabla_A \nu_{C'} = 0$$

which demands

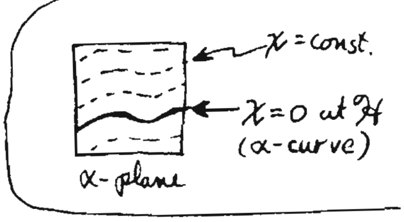
$$\Box \chi = 0$$

and which we have taken to be so. We note also that

$$X[\nu] = \chi - \nu_{A'}\pi^{A'} \in \mathbb{C}$$

is constant on the α-plane, so we have a bundle map

$$X : \mathcal{B} \longrightarrow \mathcal{T} \times \mathbb{C}.$$



This bundle map leads to the exact sequence of affine bundles

$$0 \longrightarrow \mathcal{B}_0 \longrightarrow \mathcal{B} \xrightarrow{X} \mathcal{O} \longrightarrow 0. \qquad \ldots (D)$$

The affine-bundle $\mathcal{B}_0$ is the zero set of X and encodes the cohomology class of χ alone, and the vector bundle $\hat{\mathcal{B}}$, the underlying translation bundle, or bundle of displacements of $\mathcal{B}$, encodes Ω alone.

The line bundle $\mathcal{B}_0$ has $\mathcal{O}(-2)$ as its bundle of displacements, so that (D) leads to an exact sequence of vector bundles

$$0 \longrightarrow \mathcal{O}(-2) \longrightarrow \hat{\mathcal{B}} \longrightarrow \mathcal{O} \longrightarrow 0, \qquad \ldots (E)$$

where points of the fibre of $\hat{\mathcal{B}}$ are differences between solutions of (C), i.e. solutions of $\nabla_A \nu_{A'} = 0$ on $\mathcal{M}(\mathcal{H})$. This codes the needed information of the primed spin connection. The cohomology class of $\Omega \in H^1(\mathcal{O}(-2))$ is the extension class that determines the sequence (E).

The cohomology class of $\chi \in H^1(\mathcal{O}(-2))$ determines $\mathcal{B}_0$ (see M.G.E. in TN 14 "The Penrose Transform without Cohomology"). Geometrically $\mathcal{B}_0$ encodes the location of $\mathcal{H}$ in $\mathcal{M}(\mathcal{H})$ as follows. If $\mathcal{L}$ is a holomorphic curve in $\mathbb{P}\mathcal{T}$ corresponding to a point L in $\mathcal{M}(\mathcal{H})$, a section $\mathcal{S}$ of $\mathcal{B}$ over $\mathcal{L}$ is a $\nu_{A'}(\pi_{B'})$ at L with homogeneity -1 in $\pi_{B'}$. The spinor $\nu_{A'}$ must therefore vanish and so X will map all of $\mathcal{S}$ to the value of χ in $\mathbb{C}$, which is 0 iff $L \in \mathcal{H}$. The ~~total~~ space of $\mathcal{B}$ can be canonically identified with the spin bundle of $\mathcal{M}(\mathcal{H})$ and $\mathcal{B}_0$ with its restriction to $\mathcal{H}$.

How does Einstein's equation fit into all this? How do we formulate the constraint equations and evolution equations? The following curious fact may be useful. Let $(\rho^A, \sigma_{A'})$ be a local twistor which is constant on the α-plane (consistent because $\tilde{\Psi}_{A'B'C'D'} = 0$). We have

$$\left.\begin{aligned}\nabla_A \rho^B &= -i\pi^{A'}\sigma_{A'}\,\varepsilon_A{}^B \\ \nabla_A \sigma_{B'} &= -i P_{ABA'B'}\pi^{A'}\rho^B\end{aligned}\right\}$$ on an α-plane in $\mathcal{M}(\mathcal{H})$ (N.B: $P_{ABA'B'} = \Phi_{ABA'B'}$ on $\mathcal{M}(\mathcal{H})$) $= P_{(AB)(A'B')}$

(These equations of local twistor transport are the Ward transform of $T^*(-1)\mathcal{T}$, see C.R.LeB. in Class. & Quant. Grav. 1986 "Thickenings and Gauge fields, and N.M.J.W. in " " " " 1985 "Real methods in twistor theory.) Consider $\tilde{\sigma}_A = t_A{}^{A'}\sigma_{A'}$ (with t_a not necessarily normalized); then we have

$$\nabla^A \tilde{\sigma}_A = 0 \quad \text{at } \chi = 0 \text{ on } \mathcal{M}(\mathcal{H}) \qquad \text{(F)}$$

by virtue of $\Box\chi = 0$ and of

$$P_{ABA'B'}\,t^{AA'} = 0 \quad \text{at } \chi = 0 \text{ on } \mathcal{M}(\mathcal{H}). \qquad \text{(G)}$$

Equation (G) states the <u>constraint equations</u> for Einstein's theory, the $P_{ABA'B'}$ for $\mathcal{M}(\mathcal{H})$, at $\mathcal{H}$, being a linear combination of

$$\underset{\bullet}{\Psi}_{ABCD}\,t^C_{A'}\,t^D_{B'}\,,\quad \underset{\bullet}{\tilde{\Psi}}_{A'B'C'D'}\,t^{C'}_A\,t^{D'}_B\,,\quad \underset{\bullet}{\Phi}_{ABA'B'}\,,\quad \underset{\bullet}{\Lambda}\,t_{AA'}t_{BB'}$$

where the curvature quantities with dots ($\bullet$) beneath them denote the <u>physical</u> quantities for $\mathcal{M}$. (Before, we had been using the quantities for $\mathcal{M}(\mathcal{H})$.) The terms in $\underset{\bullet}{\Psi}_{ABCD}$ and $\underset{\bullet}{\tilde{\Psi}}_{A'B'C'D'}$ disappear, in $P_{ABA'B'}\,t^{AA'}$, and we are left only with the Ricci tensor parts (because of the symmetry of $\underset{\bullet}{\Psi}_{ABCD}$, $\underset{\bullet}{\tilde{\Psi}}_{A'B'C'D'}$). Thus, by (F),

$$\tilde{\sigma}_A = \nabla_A \xi \quad \text{on } \chi = 0 \ (\text{in } \mathcal{M}(\mathcal{H})), \text{ for some } \xi$$

states the constraint equations for the Einstein theory!

What is ξ? We can find a "potential" for $P_{ABA'B'}$ at $\chi = 0$ (on an α-plane in $\mathcal{M}(\mathcal{H})$) by solving

$$\left\{\nabla_A\nabla_B + \pi^{A'}\pi^{B'}P_{ABA'B'}\right\}\eta = P_{ABA'B'}\,\nu^{A'}\pi^{B'}$$

and then we find that we can put

$$\xi = \sigma_{A'}(\nu^{A'} - \eta\pi^{A'}) + \rho^A(i\nabla_A\eta) = \sigma_{A'}\mu^{A'} + \rho^A\lambda_A$$

$$= S^\alpha W_\alpha \qquad \text{where } \begin{cases} S^\alpha = (\rho^A, \sigma_{A'}) \\ W_\alpha = (i\nabla_A\eta,\ \nu^{A'} - \eta\pi^{A'}) = (\lambda_A, \mu^{A'}) \end{cases}$$

on the α-plane in $\mathcal{M}(\mathcal{H})$.

Here W_α is <u>not</u> local twistor constant on the α-plane because of

$$\nabla_A \mu^{B'} = i\lambda_A \pi^{B'} + \nabla_A{}^{B'}\chi$$
$$\nabla_A \lambda_B = i P_{ABA'B'}\mu^{A'}\pi^{B'}$$

By Einstein (constraint equs.): integrable to 1st order away from $\chi = 0$.

This suggests that we do the following: instead of using $\mathcal{B}$, why not use $\mathcal{A}$, where the fibre over a pt. Z^α of $\mathcal{T}$ is a $W_\alpha = (\lambda_A, \mu^{A'})$ satisfying these two equations (or rather, these equs. contracted with π^A). This is an "affinized" version of the cotangent bundle $T^*(-1)\mathcal{T}$, of $\mathcal{T}$, referred to above, and we can construct the bundle of displacements $\hat{\mathcal{A}}$ of $\mathcal{A}$ (as with $\hat{\mathcal{B}}$ above) so that $\hat{\mathcal{A}} = T^*(-1)\mathcal{T}$. (Thus the $\nabla_A{}^{B'}\chi$ is removed in the passage to $\hat{\mathcal{A}}$, and $\hat{\mathcal{A}}$ <u>is</u> the cotangent bundle.) As with $\mathcal{B}$, we have a bundle map

$$X : \mathcal{A} \longrightarrow \mathcal{T} \times \mathbb{C}$$

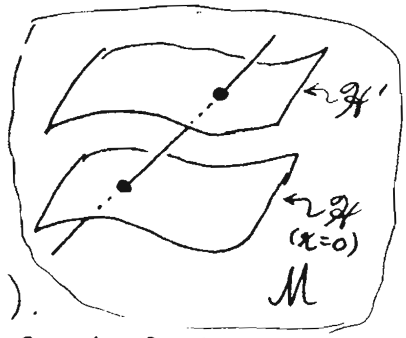
where

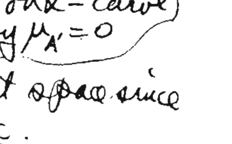
$$X(W_\alpha) = \chi + \mu^{A'}\pi_{A'} = \chi + W_\alpha Z^\alpha.$$

Z^α rep. by $(0, \pi_{A'})$ on α-plane

(On $\hat{\mathcal{A}}$ this gives $X = W_\alpha Z^\alpha$.) As with $\mathcal{B}$, $\mathcal{A}$ serves to define $\mathcal{H}$ within $M(\mathcal{H})$, and we believe that $\mathcal{A}$ fixes Ω also (not yet checked!)

What we should <u>like</u> to do would be to <u>evolve</u> the Einstein equations and obtain M. This would require something like using null geodesics defined by $W_\alpha Z^\alpha = 0$ (ambitwistors) to map from one $\mathcal{H}$ to another $\mathcal{H}'$ and to extend <u>away</u> from $W_\alpha Z^\alpha = 0$ (preferably to all orders, in some way). This would require our being able, given (W_α, Z^α) on $\mathcal{H}$, to find (W_α, Z^α) on $\mathcal{H}'$. Take $\mathcal{H}'$ to be infinitesimally above $\mathcal{H}$. What we need is a way of moving an α-curve from $\mathcal{H}$ to $\mathcal{H}'$, given a W_α on it. In the <u>flat</u> case, this movement is just that given by the α-plane in M (which now <u>exists</u>). The displacement is given by

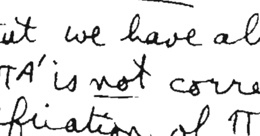




If $W_\alpha Z^\alpha = 0$, we could start off the null geodesic defined by (W_α, Z^α) at the point on the α-curve on $\mathcal{H}$ where $\mu^{A'} = 0$, and then displace away along $\lambda^A \pi^{A'}$. We want this displacement to work also for the <u>other</u> points on the α-curve. This is O.K. in flat space since we can take the direction of $\lambda^A\pi^{A'}$ elsewhere on the α-curve to be <u>constant</u>.

connecting vectors (null) $\lambda_\times^A \pi_\times^{A'}$

$t_{AA'}\lambda_\times^A \pi_\times^{A'} = t_{AA'}t^{AA'}$



original pt. on α-curve on ℋ given by $\mu_{A'} = 0$

In curved space it seems possible that the λ^A that we have already been using might still work (i.e. $\lambda_\times^A = \lambda^A$), but the $\pi^{A'}$ is <u>not</u> correct (i.e. $\pi_\times^{A'} \neq \pi^{A'}$, in general). We seem to need a modification of $\pi^{A'}$ in order for this to work elsewhere along the curve. So far we do not have a suitable primed spinor that varies along the α-curve in a way that is not proportional to $\pi^{A'}$. We need all this so that we can propagate (W_α, Z^α) when $W_\alpha Z^\alpha \neq 0$, and there is no null geodesic defined by (W_α, Z^α).

More work in progress.

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