An alternative form of the Ernst potential

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One of the strengths of the twistor description of the stationary axisymmetric solutions of Einstein's equations is the light it sheds on the relation between the metric and its Ernst potential. Recall that to represent a solution we have a bundle E over a reduced twistor space R_U which consists of two Riemann spheres S_0 and S_1 (with coordinate w) identified over the set U. The usual Ward construction gives us a solution of Yang's equation, usually denoted by J, which is the metric on the space of Killing vectors. If we impose the conditions that J be regular on the symmetry axis $r = 0$ and satisfy $\det J = -r^2$ then $E|_{S_0} = L_1 \oplus L_0$ and $E|_{S_1} = L_{-1} \oplus L_0$ where L_1 is the tautological bundle, L_{-1} is the hyperplane section bundle and L_0 is the trivial bundle. We can describe E by means of patching matrices $P_{\alpha\beta}$ defined on the overlaps of a collection of open sets U_α which cover R_U. If we take $U_0 \subset S_0$ and $U_1 \subset S_1$ to be neighbourhoods of $w = \infty$ not containing $w = 0$, and $U_2 \subset S_0$ and $U_3 \subset S_1$ to be neighbourhoods of $w = 0$ not containing $w = \infty$, then

$$P_{02} = \begin{pmatrix} 2w & 0 \\ 0 & 1 \end{pmatrix}, \quad P_{13} = \begin{pmatrix} (2w)^{-1} & 0 \\ 0 & 1 \end{pmatrix}$$

and E is completely specified by one of the patching matrices between the spheres, for example by P_{23} which I shall denote by P.

We can construct another solution $\iota(J)$ from the related bundle $\iota(E)$. To obtain $\iota(E)$, we take the same cover $\{U_\alpha\}$ and the same patching matrix P between the two spheres, but replace both P_{02} and P_{13} by the identity. Thus the restrictions $\iota(E)|_{S_0}$ and $\iota(E)|_{S_1}$ are both trivial; and we can think of the operations ι and ι^{-1} as untwisting and twisting the bundle round the points at infinity. We can write the corresponding matrix in the form

$$\iota(J) = \begin{pmatrix} f - \frac{\psi^2}{f} & \frac{\psi}{f} \\ \frac{\psi}{f} & \frac{1}{f} \end{pmatrix}$$

then the usual Ernst potential $\mathcal{E}$ is given by $\mathcal{E} = f + i\psi$. NMJW and LJM (1988) used this as the starting point for encoding the Geroch group in the twistor picture.

Instead of twisting and untwisting about $w = \infty$, however, we can do the same about $w = 0$. In other words, given our original bundle E, we can define a new one $\hat{E}$ which is also trivial over each of the spheres S_0 and S_1, but for which the patching matrix between them is P_{01} instead of P_{23}. Since P_{01} is given by

$$P_{01} = \begin{pmatrix} 2w & 0 \\ 0 & 1 \end{pmatrix} P \begin{pmatrix} 2w & 0 \\ 0 & 1 \end{pmatrix},$$

it has determinant equal to $(2w)^2$ and it is actually more convenient to use $P' = (1/2w)P_{01}$ to define the new bundle E'. I explained in my article in TN27 what it means for a patching matrix to be *adapted* to a certain part of the axis $r = 0$; if we suppose that P is adapted to an interval of the form $(0, a)$ then replacing P_{01} with P' corresponds to dividing the corresponding matrix J by u^2 to obtain J', where (in terms of the usual Weyl coordinates (z, r)), u and v are given by $r = uv$ and $z = \frac{1}{2}(u^2 - v^2)$.

Note that there is a certain amount of freedom in the construction of J'. With a particular choice of the two Killing vectors (X_1, X_2) in the original space-time (which must be arranged such that X_1 vanishes or is null on $r = 0$ in the interval $(0, a)$), we can still transform P by $P \mapsto BPC$, where

$$B = \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix}$$

for constants b and c. This will alter J' while leaving J the same. Going in the other direction, however, there is a unique P' for each J' and the only choice that can arise occurs if the twistor space $R_{U'}$ is glued down at $w = 0$; we then have to decide to which of the spheres S_0 and S_1 to assign each of the points at $w = 0$ in R_U.

There is a direct method for passing between J' and J which is analogous to changing from Ernst potential to metric. If we choose a Ward splitting $\{K'_\alpha\}$ for the patching matrices $P'_{\alpha\beta}$ describing E' such that

$$K'_2 = \begin{pmatrix} * & 0 \\ * & 1 \end{pmatrix} \text{ at } \lambda = -(v/u) \text{ and } K'_3 = \begin{pmatrix} 1 & * \\ 0 & * \end{pmatrix} \text{ at } \lambda = u/v$$

(where λ is the coordinate on the CP^1 in PT corresponding to the orbit (u, v)) then, provided we have chosen S_0 and S_1 such that $w(u/v) \in S_0$ and $w(-v/u) \in S_1$, it follows that

$$J = H' \begin{pmatrix} -v^2 & 0 \\ 0 & u^2 \end{pmatrix} (\hat{H}')^{-1}$$

where $H' = K'_0(0)$ and $\hat{H}' = K'_1(\infty)$ (and so $J' = H'(\hat{H}')^{-1}$).

This choice of K'_2 and K'_3 corresponds to a choice of complex structure Ψ (in the Dolbeault version of the Ward construction: see NMJW & LJM (1986) such that

$$\Psi \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 0 \text{ at } \lambda = u/v \text{ and } \Psi \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0 \text{ at } \lambda = -v/u. \qquad (1)$$

This is because, on the pull-back of the set U_α to PT, $\Psi = K_\alpha^{-1}\partial K_\alpha$.

It is straightforward to show (from the definition of Ψ) that (1) implies we have chosen $\hat{H}' = (s_1\ s_2)$ and $H' = J'\hat{H}'$, where s_1 and s_2 are solutions of the equations

$$\partial_r s + \frac{1}{1+\lambda^2}((J')^{-1}\partial_r J' - \lambda (J')^{-1}\partial_z J')s = 0 \tag{2}$$

$$\partial_z s + \frac{1}{1+\lambda^2}(\lambda (J')^{-1}\partial_r J' + (J')^{-1}\partial_z J')s = 0 \tag{3}$$

with $\lambda = u/v$ and $\lambda = -v/u$ respectively. To go in the other direction, we replace J' with $u^{-2}J$ and choose $\hat{H} = (s_2\ s_1)$.

There are two dimensions of freedom in the choice of s_1 and s_2 in each case; and in fact defining $\hat{H}'$ in this way only implies that K'_2 and K'_3 are of the form

$$K'_2 = \begin{pmatrix} * & \alpha \\ * & \beta \end{pmatrix} \quad \text{and} \quad K'_3 = \begin{pmatrix} \gamma & * \\ \delta & * \end{pmatrix}$$

where α and β are constant on $\lambda = u/v$, and γ and δ are constant on $\lambda = -v/u$. Defining $\hat{H}$ in a similar way gives a similar form for K_2 and K_3. In both cases, the behaviour of the splitting matrices on the two surfaces in PT is due to the fact that the only holomorphic functions on PT which are invariant under the lifts of the two Killing vectors are those which are functions of w alone, where w is related to λ by the equation

$$w = \frac{r}{2}(\lambda^{-1} - \lambda) + z.2) \tag{4}$$

When we go in the 'twisting' direction (that is to say, from J' to J) we can actually fix s_1 and s_2, and thus J, completely by considering the behaviour of K'_2 and K'_3 as $v \to 0$. This corresponds to the unique choice of P' in this case. On the other hand, the freedom that we have in the other direction also corresponds precisely to the freedom in the choice of patching matrix P.

Finally, a brief remark on the point of all this. It turns out that if a space-time has both a symmetry axis and a Killing horizon and is regular at the point where they intersect, then the patching matrix P has a simple pole at the point in the reduced twistor space which corresponds to the intersection and which we can assume to be at $w = 0$ (see JF & NMJW 1990). It is straightforward to show, however, that the 'untwisted' patching matrix, P', is well-behaved on the real axis near $w = 0$, and slightly less straightforward to show that its entries are actually holomorphic in a neighbourhood of this point.

References

NMJW & LJM Nonlinearity **1** 73 - 114 (1988) JF in TN **27** 14 - 16 JF & NMJW in 'Reviews in Twistor Theory' ed RJB & TNB (1990)

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