Structure of the jet bundle for manifolds with conformal or projective structure

Let G be a complex semi-simple Lie group and P<G a Lie subgroup. To each P-module E there corresponds a homogeneous vector bundle $\boldsymbol{E}$, over G/P, which is just $\mathrm{G}\times E$ modulo the equivalence relation $(\mathrm{g},e) \sim (\mathrm{gp}, \mathrm{p}^{-1}e)$ for g∈G,p∈P and $e \in E$. We write

$$\boldsymbol{E} = \mathrm{G}\times_{\mathrm{P}} E.$$

It is an interesting exercise (see [2]) to check that the the bundle of infinite jets, $\boldsymbol{JE}$, associated with this homogeneous bundle is itself homogeneous:

$$\boldsymbol{JE} = \mathrm{G}\times_{\mathrm{P}} JE$$

where JE may be obtained as the fibre of $\boldsymbol{JE}$ over the identity coset P∈G/P.

It is interesting and useful to see the extent to which this goes through in case G/P is replaced by a more general structure. So let $\mathcal{M}$ be an arbitrary complex holomorphic manifold equipped with a P-principal bundle $\mathcal{G}$. Then $\mathcal{M}$ is the fibrewise quotient $\mathcal{G}/\mathrm{P}$ and we have an appropriate generalisation of the above situation. Corresponding to the homogeneous bundles above are *semi-homogeneous* bundles which are constructed in exactly the same way: If E a P-module then we have $\boldsymbol{E} := \mathcal{G} \times_{\mathrm{P}} E$.

A section f of $\boldsymbol{E}$ corresponds to a function F,

$$F : \mathcal{G} \to E,$$

such that

$$\mathrm{p}F(g\mathrm{p}) = F(g) \quad \text{for} \quad g \in \mathcal{G} \text{ and } \mathrm{p} \in \mathrm{P}.$$

Such functions F will be said to be *semi-homogeneous*. The space of such E-valued semi-homogeneous functions is itself a P-module in the obvious way. This has a P-submodule of semi-homogeneous functions which vanish to order $k+1$ on $g\mathrm{P}$ for $x = g\mathrm{P}$ an arbitrary point of $\mathcal{M}$. The quotient P-module shall be denoted $J^k_x E$. Points of the bundle of k-jets associated to $\boldsymbol{E}$, over x, correspond to points in $g\mathrm{P}\times J^k_x E$ modulo the equivalence relation $(g, F) \sim (g\mathrm{p}, \mathrm{p}^{-1}F)$ for $\mathrm{p} \in \mathrm{P}$ and $F \in J^k_x E$. Thus we see that the bundle of k-jets, $\boldsymbol{J^k E}$, has an underlying P-structure as in the homogeneous case above. However in this more general setting the inducing P-module may vary from point to point of $\mathcal{M}$. We could write

$$\boldsymbol{J^k E} = \mathcal{G} \times_{\mathrm{P}} J^k_{g\mathrm{P}} E$$

to describe this. With the same notation the bundle of infinite jets is given by

$$\boldsymbol{JE} = \mathcal{G} \times_{\mathrm{P}} J_{g\mathrm{P}} E,$$

where, for each $g\mathrm{P} \in \mathcal{G}/\mathrm{P}$, $J_{g\mathrm{P}}E$ is the projective limit over k of the $J^k_{g\mathrm{P}}E$.

The dual version of this proceeds as follows. Let $\mathcal{D}$ denote the space of differential operators from E-valued functions on $\mathcal{G}$ to $\mathbf{C}$-valued functions on $\mathcal{G}$. When restricted to act on semi-homogeneous functions, F, $\mathcal{D}$ gains a P-module structure: For $D \in \mathcal{D}$, $\mathrm{p}D$ is defined by

$$[\mathrm{p}D]F(g) := DF(\tilde{g})|_{\tilde{g}=g\mathrm{p}} = D\mathrm{p}^{-1}F(\tilde{g}\mathrm{p}^{-1})$$

where $g \in \mathcal{G}$, p$\in$P and on the extreme right hand side $F(\tilde{g}\mathrm{p}^{-1}) = F(g)$ is to be regarded as a function of $\tilde{g}$. $\mathcal{D}$ has a P-submodule of operators which act as zero on semi-homogeneous functions. Denote by $\tilde{X}(E)$ the quotient P-module. Let $\mathcal{O}_{\mathcal{M}}$ denote the C-valued semi-homogeneous functions on $\mathcal{G}$, i.e., the functions constant on each fibre gP, $g \in \mathcal{G}$. For some fixed $x \in \mathcal{M}$, let $\mathcal{I}_x < \mathcal{O}_{\mathcal{M}}$ be the subspace of functions which vanish over x. Denote by $\mathcal{I}_x.\tilde{X}(E)$ the P-submodule of $\tilde{X}(E)$ which consists of elements of $\tilde{X}(E)$ left multiplied by functions in $\mathcal{I}_x$. Since $\tilde{X}(E)$ is naturally a $\mathcal{O}_{\mathcal{M}}$-module it is clear that $\mathcal{I}_x.\tilde{X}(E)$ is a P-submodule of $\tilde{X}(E)$. Once again we can form the quotient module which we shall denote $\tilde{V}_x(E)$. This P-module is filtered naturally by order of the operators involved; write $\tilde{V}_{k,x}(E)$ to denote the submodule consisting of operators of order no greater than k. Since each element of $\tilde{V}_{k,x}(E)$ determines a map

$$J^k E \to \mathbf{C},$$

and $\tilde{X}(E)$ consists of all non-trivial operators on E-valued semi-homogeneous functions it is at once clear that $\tilde{V}_{k,x}(E)$ is precisely the vector dual of $J^k_x E$. It is easily checked that it is also dual as a P-module. Evidently then,

$$(\boldsymbol{J}^k \boldsymbol{E})^* = \mathcal{G} \times_{\mathrm{P}} \tilde{V}_{k,g\mathrm{P}}(E)$$

with notation understood to be as above.

While this and the dual version first mentioned provide a description of the jet bundle the situation is less than ideal. Even at the level of k-jets, since the inducing module is point dependent there is little scope for reducing the problem of finding differential operators of order $\leq k$ to a finite dimensional one. Nevertheless without more structure this is probably as far as one can go. However in many instances such structure is readily available ....

For example if $\mathcal{M}^n$ has a conformal (or projective) structure then one obtains a principal P-bundle, $\mathcal{G}$, where P is a particular parabolic subgroup of Spin$(n+2)$ (SL$(n+1)$ respectively). Moreover the bundle $\mathcal{G}$ comes equipped with a canonical notion of horizontality called the normal conformal (resp. projective) Cartan connection. We shall see that in either of these cases the jet bundle is almost as simple to describe as in the homogeneous case. The Cartan connection (which will always refer to the *normal* version) is usually described by a 1-form ϑ which satisfies (where $\mathbf{p}$ and $\mathbf{g}$ are the Lie algebras of P and G respectively):

(i) $\vartheta_q : T_q\mathcal{G} \to \mathbf{g}$ is an vector space isomorphism $\forall q \in \mathcal{G}$.

(ii) $\vartheta(\mathrm{X}^*_q) = \mathrm{X}$ if X^* is the Killing field corresponding to X$\in \mathbf{p}$.

(iii) $R^*_{\mathrm{p}}\vartheta = \mathrm{Ad}(\mathrm{p}^{-1})\vartheta$, where R_{p} describes the right action of p$\in$P on $\mathcal{G}$.

as well as some curvature conditions. Note that if we write $\mathbf{g}^* := \vartheta^{-1}(\mathbf{g})$ then, regarding the vector fields $\mathbf{g}^*$ as differential operators, (iii) is equivalent to

$$[\mathrm{XY}]^* = [\mathrm{X}^*, \mathrm{Y}^*]$$

for arbitrary X$\in \mathbf{p}$ and Y$\in \mathbf{g}$. We can, in the obvious way, extend ϑ^{-1} to act on the tensor algebra, $\bigotimes \mathbf{g} := \bigoplus_{k=0}^{\infty} \otimes^k \mathbf{g}$. The result of this is a space of special differential operators on $\mathcal{G}$ which will be denoted by $\mathcal{U}(\mathbf{g}^*)$. There is a natural filtration of $\mathcal{U}(\mathbf{g}^*)$ induced from the grading of the tensor algebra $\bigotimes \mathbf{g}$; i.e., $\mathcal{U}_k(\mathbf{g}^*)$ is the image of $\bigoplus_{i=0}^k \otimes^i \mathbf{g}$. We note that $\mathcal{U}(\mathbf{g}^*)$ is strictly contained in $\mathcal{D}$, in fact $\mathcal{U}_k(\mathbf{g}^*)$ is finite dimensional.

The left $\mathcal{U}(\mathbf{g}^*)$-module

$$\mathcal{U}(\mathbf{g}^*) \otimes E^*$$

may be thought of as a special class of differential opertors from E-valued functions on $\mathcal{G}$ to $\mathbf{C}$-valued functions on $\mathcal{G}$. As operators restricted to semi-homogeneous functions we may consider the action of P (as described above for all of $\mathcal{D}$) on this space. Now we may regard this $\mathcal{U}(\mathbf{g}^*)$-module as a $\mathbf{p}^*$-module (or equivalently a $\mathbf{p}$-module) by restriction and it is readily verified that this agrees precisely with the P-action (at least treating the elements of $\mathcal{U}(\mathbf{g}^*) \otimes E^*$ as differential operators on semi-homogeneous functions). Thus $\mathcal{U}(\mathbf{g}^*) \otimes E^*$ is closed under this P-action and so, given property (i) of ϑ, is an ideal candidate to replace $\mathcal{D}$.

$\mathcal{U}(\mathbf{g}^*) \otimes E^*$ has a P-submodule of operators which annihilate all semi-homogeneous functions. Let $X(E)$ be the quotient and $\mathcal{O}_{\mathcal{M}}.X(E)$ consist of elements of $X(E)$ left multiplied by functions from $\mathcal{O}_{\mathcal{M}}$. Then $\mathcal{O}_{\mathcal{M}}.X(E)$ is also a P-module and, for any fixed $x \in \mathcal{M}$, has $\mathcal{I}_x.X(E)$ as a P-submodule. Again we form the quotient and denote the resulting P-module $V_x(E)$. With similar reasoning to that in used in the $\tilde{V}_x(E)$ case it is not difficult to see that $V_x(E) = (J_x E)^*$ and that

$$\mathcal{G} \times_{\mathrm{P}} V_{gP} E \equiv (\boldsymbol{J} \boldsymbol{E})^*.$$

In this construction also, the inducing P-module varies on $\mathcal{M}$. Thus at first glance it would seem that we are no better off than with the construction that began with $\mathcal{D}$. In fact, however, we now have a considerably more rigid structure as consideration at the level of k-jets reveals.

Write $\mathcal{D}_k < \mathcal{D}$ to mean the subspace of differential operators of order $\leq k$. Corresponding to this $\tilde{X}(E)$ will inherit a filtration, by $\tilde{X}_k(E)$ say. In the approach that begins with all differential operators, this is the key P-module leading to the construction of the dual k-jet bundle. The problem is that $\tilde{X}_k(E)$ is infinite dimensional and we know nothing about its structure. If a conformal or projective structure is present then corresponding to this one has $X_k(E)$, where the filtration of $X(E)$ by the $X_k(E)$ arises from the filtration of $\mathcal{U}(\mathbf{g}^*)$ by $\mathcal{U}_k(\mathbf{g}^*)$. Now, in contrast to $\tilde{X}_k(E)$, $X_k(E)$ is finite dimensional. In fact $X_k(E)$ looks just like certain Verma modules which arise in the homogeneous case with some modification due to the curvature of the Cartan connection. Thus, although the structure of $V_{k,x}(E)$ varies over $\mathcal{M}$, the variation involved is a relatively minor detail involving the actual value of the curvature at each point. The important point, however, is that we have a natural bundle epimorphism from a finite dimensional semi-homogeneous bundle onto $(\boldsymbol{J}^k \boldsymbol{E})^*$:

$$\mathcal{G} \times_{\mathrm{P}} X_k(E) \to (\boldsymbol{J}^k \boldsymbol{E})^*.$$

There is an immediate application of this result. Suppose there is a P-module monomorphism

$$i : H^* \to X_k(E).$$

Then this induces an invariant homomorphism of the corresponding semi-homogeneous bundles:

$$\mathcal{G} \times_{\mathrm{P}} H^* \to \mathcal{G} \times_{\mathrm{P}} X_k(E).$$

and thus a vector bundle homorphism,

$$\mathcal{G} \times_{\mathrm{P}} H^* \to (J^k E)^*.$$

Dually then, we have a bundle homomorphism

$$\boldsymbol{J^k E \to H},$$

that is, a differential operator; here $\boldsymbol{H}$ is of course $\mathcal{G} \times_{\mathrm{P}} H$. Beginning with irreducible modules H, finding injections such as i above is straightforward (in principle at least) and just involves finding certain vectors in $X(E)$ which are annihilated by a special subalgebra of $\mathbf{p}$. (These are called *maximal* vectors.) It is thus easy to see that for irreducible H and any k there are a finite number of differential operators which arise in this fashion. Indeed beginning also with E irreducible, the resulting operators are, in a real sense, analogues of the *invariant* operators in the homogeneous case (as in [2,1]) or composites thereof.

Examples of applications of these ideas can be found in [3].

References

[1] R. J. Baston. *The Algebraic Construction of Invariant Differential Operators.* D.Phil. thesis. Oxford University (1985).

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[3] A. R. Gover. *A Geometrical Construction of Conformally Invariant Differential Operators.* D.Phil. thesis. Oxford University (1989).

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